The inefficiency of
Constant Proportion Portfolio Insurance

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Introduction

• Focal point: Sub-optimality of the path-dependent investment strategies

• The presentation shows ideas developed in:
Outline of the talk

- Recap on CPPI
- Research problem:
  *Are the path-dependent investment strategies sub-optimal?*
- Our findings
- Numerical examples
- Conclusions
- References
Introduction to CPPI

**CPPI**: Constant Proportion Portfolio Insurance

- is a dynamic portfolio insurance strategy that aims to protect the investor against adverse market conditions (market risk)

- is a generalization of stop-loss and buy and hold strategies and allows one to tune the risk profile between these two strategies

- is based on managing the balance of assets between the risky assets (typically some form of equity underlying) and the riskless assets (typically a zero-coupon bond)

- is similar to delta hedging of a short gamma position: in a falling market the underlying assets are being sold and in rising markets the underlying assets are being bought
CPPI Vocabulary

CPPI is a formula driven strategy:
Participation = Multiplier x Equity Gap

- Exposure $E(t)$
- Equity gap - Cushion $C(t)$
- Multiplier - Leverage factor $m$
- Bond Value - Floor $F(t)$
The CPPI Algorithm

1. Specify starting parameters: \( \text{NAV}(0), F, m, T, \Delta T \) and risk-free rate \( r \)
2. Calculate \( C(t) \), the difference between the net-asset-value (NAV) of the overall investment and the present value (PV) of the capital guarantee according to:

\[
C(t) = PV(t) - F(t),
\]

where floor grows with the risk-free interest rate \( r \): \( dF(t) = rF(t)\,dt \)
3. Calculate \( E(t) \), the exposure to the risky assets, according to:

\[
E(t) = m \times C(t)
\]

4. If \( E(t) \) is positive, then invest \( E(t) \) in the risky assets and the residual amount in the risk-free assets
5. Wait until \( \Delta T \) has passed and go back to step 2
6. Repeat steps 2-5 until the end of the investment horizon \( T \)
Plain Vanilla CPPI: Example

- **Scenario 1** (Bullish market)
  Investors benefit from upside offered by the structure and enjoy the bullish market

- **Scenario 2** (Bearish market)
  Investors benefit from security offered by the structure and are guaranteed to recover the full principal at maturity
Pros

- Allows one to match any risk appetite by choosing an appropriate multiplier and target level. It is also possible to change dynamically multiplier in accordance with the risky asset volatility in order to obtain a desired portfolio performance.

- Yield enhancement with principal protection (suitable for pension funds)

- Reduction of regulatory capital requirement

- Duration extension (suitable for LDI)

- It can be applied to different assets classes: equity, fixed income, commodities, structured products, e.g. CDO equity, hedge funds or funds of funds, long/short combinations.
Cons

- Transaction costs
- Gap Risk
- Path-Dependence
The financial market is frictionless, trading is continuous, and there exists a constant risk-free interest rate $r > 0$. There are no taxes, no dividends, no restrictions on borrowing or short sales.

- $\mathbb{P}$ denotes the physical (historical) probability measure.

- We consider an investor who is facing a fixed investment horizon of length $T > 0$, hence she/he cares only about the final payoffs and do not care about intermediate values.
Notation cont’d

- Financial security with pay-off $P_g$ at time $t = T$:

$$P_g = g(S(t_i) \mid 0 \leq t_i \leq T, \ i = 1, 2, \ldots, n),$$

where $S(t_i)$ is the price of the risky asset at time $t_i > 0$.

- We consider an arbitrage-free market and there exist risk neutral measure $\mathbb{Q}$ employed for obtaining the price/cost of the strategy:

$$C(P_g) = e^{-rT}E[P_g]$$

- If $P_g$ depends on the final asset value $S(T)$ only, then $P_g$ is path-independent. Otherwise we call it a path-dependent pay-off.

- Examples of path-dependent pay-offs include Asian Options, ‘Clickfunds’ and all ”CPPI - like investments”.

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Lévy processes

- We say \( \{X_t, t \geq 0\} \) is a Lévy process if
  - \( X_0 = 0 \)
  - stationary and independent increments and \( X_t \) follows an infinitely divisible distribution
- Lévy-Khintchine formula:

\[
\log E(\exp(iuX_1)) = i\gamma u - \frac{\sigma^2}{2} u^2 + \int_{-\infty}^{\infty} (\exp(iux) - 1 - iux I_{\{|x|<1\}}) \nu(dx),
\]

where \( \gamma \in \mathbb{R}, \sigma^2 \geq 0 \) and \( \nu \) is a measure on \( \mathbb{R}/\{0\} \) with \( \int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty \)

- In order to obtain the risk neutral measure we employ the Esscher transform with parameter \( h \in \mathbb{R} \). This transformation of a stochastic process \( \{X_t, t \geq 0\} \) is the process where for \( t > 0 \) the modified probability density function \( f_t^{(h)}(x) \) of \( X_t \) is defined as:

\[
f_t^{(h)}(x) = \frac{e^{hx} f_t(x)}{m_t(h)}
\]
Research Questions

• Can a given $P_g$ be dominated from the point of view of risk averse decision makers with a fixed investment horizon $T$?

• Is it possible to find the optimal investment strategy within the class of dominating pay-offs $P_g$?
Cox and Leland (1982) found that in the particular case of a Geometric Brownian Motion for \( \{S(t)\}_{t \geq 0} \), risk averse decision makers with a fixed investment horizon \( T \) prefer path-independent pay-offs over path-dependent pay-offs.

We generalise Cox and Leland’s results to general Geometric Lévy processes for \( \{S(t)\}_{t \geq 0} \).

These results mean that all path-dependent strategies are in general value destroying.

Furthermore, we show that for fixed horizon investors, optimal path-independent pay-offs are those that increase with the underlying asset value. This finding is closely related to results of Dybvig (1988a,b) and is based on two observations:
Observation no.1

Classical von Neuman & Morgenster Utility Theory

- We prefer wealth $Y$ over wealth $X$ if
  \[ E[u(X)] \leq E[u(Y)], \]
  where $u(x)$ is an increasing, convex utility function.

- Let us now ‘approximate’ $P_g$ by $E_P[P_g \mid S(T)]$
  \[ P_g = E_P[P_g \mid S(T)] + \varepsilon \]

- From the Jensen’s inequality it follows that
  \[ E_P[P_g \mid S(T)] \leq_{\text{cxt}} P_g \]

- Therefore, provided the prices (obtained by arbitrage free pricing) for these pay-offs are the same, the pay-off $E_P[P_g \mid S(T)]$ will dominate the pay-off $P_g$ from the point of view of all risk averse decision makers.
Observation no.2

- We use the Esscher transform to derive the risk-neutral measure $Q$ in case of Geometric Lévy Processes; see Gerber and Shiu (1994)

- The Esscher transform is desirable since it preserves the same structure of Lévy process under the risk-neutral measure as that under the objective measures

- Now it is easy to prove that:

$$E_P[P_g | S(T)] = E_Q[P_g | S(T)]$$

because the conditional densities are the same under $Q$ as under $P$, when conditioning on $S(T)$

- It follows that:

$$E_Q[P_g] = E_Q[E_Q[P_g | S(T)]] = E_Q[E_P[P_g | S(T)]]$$
Equality of Conditional Densities

• We have to prove that for all real \( x, y \) and \( t \geq 0 \) we have:

\[
f_t(x \mid X(T) = y) = f^h_t(x \mid X(T) = y),
\]

where \( \{X(t)\}_{t\geq0} \) is a Lévy process and \( f^h_t(x) \) is denoting the density under the Esscher equivalent martingale measure

• Proof:

\[
f^h_t(x \mid X(T) = y) = f^h_{t-T}(x, y-x) \cdot \frac{f^h_T(y)}{f^h_T(x)} = \frac{f_t(x) \cdot e^{hx} \cdot f_{T-t}(y-x) \cdot e^{hy-hx}}{f_T(y) \cdot e^{hy}} \cdot \frac{M_{X(T)}(h)}{M_X(T) \cdot M_{X(T)-X(t)}(h)}
\]

\[
= \frac{f_t(x) \cdot f_{T-t}(y-x)}{f_T(y)} = f_t(x \mid X(T) = y)
\]
Optimal path-independent strategy

• If $E[S(t)] > e^{rt}$ then the best path-independent pay-off is when the pay-off is increasing with the final stock value at time $T$.

• The main idea being that we can keep the same physical distribution, whilst consistently assigning the largest realisations for the payoff to the largest values of the underlying stock - where the consumption is cheaper.

• Remark that it goes beyond doubt that investors when choosing between pay-offs that have the same distribution will pick the one with the lowest cost.

• **Theorem**: Assume that $\{\log(S_t), t \geq 0\}$ is a Lévy process and $\log(S_t)$ is a continuous random variable. Moreover there is a path-dependent pay-off $g(S_T)$ where $g$ is piecewise continuously differentiable but not non-decreasing. Then there exist a function $h$ such that, under $\mathbb{P}$, we have that $h(S_T) \overset{d}{=} g(S_T)$ and $C(h(S_T)) < C(g(S_T))$. 

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Geometric averaging

- Let us consider an index \( \{S_t\}_{t \geq 0} \) that is a Geometric Brownian motion with real parameters \( \mu \) and \( \sigma \). Moreover \( S_0 = 1 \) and \( T = 2 \)

- Consider a path dependent pay-off

\[
P_g = (S_1 S_2)^{1/2} = e^{Z_1 + \frac{Z_2}{2}},
\]

where the random variables \( Z_1 = X_1 \) and \( Z_2 = X_2 - X_1 \) are independent and normally \( N(\mu, \sigma^2) \) distributed log-returns over the periods \([0, 1)\) and \([1, 2)\)

- \( P_g \) is lognormally distributed with parameters \( \frac{3}{2} \mu \) and \( \frac{5}{4} \sigma^2 \)
Geometric averaging cont’d

- Consider path-independent strategy $E_P(P_g|S_2)$ is given by:

$$E_P(P_g|S_2) = E_P(P_g|X_2) = E_P(P_g|Z_1 + Z_2) = E_P(e^{Z_1 + \frac{Z_2}{2}}|Z_1 + Z_2)$$

- From the properties of the bivariate normal random vector $(Z_1 + \frac{Z_2}{2}, Z_1 + Z2)$ we find

$$E_P(P_g|S_2) = e^{\frac{3}{4}(Z_1 + Z_2) + \frac{5}{80}\sigma^2} = (S_2)^{\frac{3}{4}}e^{\frac{5}{80}\sigma^2},$$

hence $E_P(P_g|S_2)$ is lognormally distributed with parameters $\frac{3}{2}\mu + \frac{5}{80}\sigma^2$ and $\frac{9}{8}\sigma^2$

- Comparing set of parameters $(\frac{3}{2}\mu + \frac{5}{80}\sigma^2, \frac{9}{8}\sigma^2)$ with $(\frac{3}{2}\mu, \frac{5}{4}\sigma^2)$ we see that if the costs are equal the pay-off $E_P(P_g|S_2)$ will be preferred over the $P_g$ pay-off.
Geometric averaging cont’d

• Under the Esscher equivalent measure \( \mathbb{Q} \) one replaces the parameter \( \mu \) by \( (r - \frac{1}{2} \sigma^2) \)

\[
C(P_g) = e^{-2r} \cdot e^{\frac{3}{2} (r - \frac{1}{2} \sigma^2) + \frac{5}{8} \sigma^2} = e^{-\frac{1}{2} r - \frac{1}{8} \sigma^2}
\]

• We find that

\[
C(\mathbb{E}_\mathbb{P}(P_g | S_2)) = e^{-2r} \cdot e^{\frac{3}{2} (r - \frac{1}{2} \sigma^2) + \frac{5}{80} \sigma^2 + \frac{9}{16} \sigma^2} = e^{-\frac{1}{2} r - \frac{1}{8} \sigma^2}
\]

• We found dominating path-independent strategy
Click fund

- Invested amount = 100 and \( r \) is risk-free rate

- If the yearly return of the underlying market index is positive then we 'click' 10% for that year

- After 8 years the investor receives \( 100(1 + \#\text{clicks} \times 10\%) \) i.e.

\[
P_g = 100 \left( 1 + 0.1 \sum_{i=1}^{8} l_i \right)
\]

where the indicator random variable \( l_i = 1 \) if \( S_i > S_{i-1} \) and \( l_i = 0 \) otherwise.
Click fund cont’d

- Denote by $P(N(\mu, \sigma^2) > 0)$ by $p$ and $P(N(r - \frac{1}{2}\sigma^2, \sigma^2) > 0)$ by $q$
- The physical distribution of $P_g$ is following

$$P(P_g = 100 + 10i) = \binom{8}{i} p^i (1 - p)^{8-i}, \quad i = 0, 1, \ldots, 8,$$

- Denote $P(P_g \leq 100 + 10i)$ by $k_i$
- Consider another pay-off $P_h$

$$P_h = 100 \left(1 + 0.1 \sum_{i=1}^{8} J_i\right),$$

where $J_i$ is the indicator random variable that takes value 1 if $S_8 > \alpha_i$, $(i = 1, 2, \ldots, 8)$ with $\alpha_i = e^{8\mu + \sqrt{8}\sigma^2 \phi^{-1}(k_i-1)}$. $\phi$ denotes the quantile function of the standard normal random variable.
Click fund cont’d

- Under the $\mathbb{P}$ the pay-off $P_h$ has the same distribution function as $P_g$

- $P_g$ price:
  \[ C(P_g) = e^{-8r}(100 + 80q) \]

- $P_h$ price:
  \[ C(P_h) = 100e^{-8r}(1 + 0.1 \sum_{i=1}^{8} P(J_i = 1)), \]

  where $P(J_i = 1)$ denotes the probability under $\mathbb{Q}$ that $S_8 > \alpha_i$

- Market parameters: $\mu = 0.1$, $r = 0.046$, $\sigma = 0.2$

- Results: $C(P_g) > C(P_g)$ while physical distributions are the same (see Excel example)
CPPI example

- Plain vanilla CPPI strategy with restrictions (= no short positions), where floor grows with the constant risk-free rate $r > \mu$

- Initial investment : 100, Leverage factor (multiplier): $m = \{2, 4, 7\}$

- Investment horizon: 1 and 5 years, rebalanced monthly

- Minimum target level : 100% (full protection)

- Market parameters:
  - risk-free rate $r = 4\%$
  - risk premium = 4\%
  - standard deviation $\sigma = 0.1619$
CPPI example cont’d

We consider two investment universe set-up:

1. $S(t)$ is a Geometric Brownian Motion with $\mu$ and $\sigma^2$

2. $\log(S(t))$ is a Normal Inverse Gaussian process i.e.

$$\{\log(S(t)), t \geq 0\} \sim \text{NIG}(\mu t, \alpha, \beta, \delta t),$$

- with parameters

$$\mu = 0.08, \alpha = 6.1882, \beta = 0, \delta = 0.1622$$

- This corresponds to the market parameters equal to

$$\mu = 0.08, \sigma = 0.1619, \text{skewness} = 0, \text{kurtosis} = 5.9889$$
CPPI example cont’d

• Under the real-world measure returns $X_t$ follow a $NIG(\mu t, \alpha, \beta, \delta t)$

• Under the Esscher measure $\mathbb{Q}$ $X_t$ are $NIG(\mu t, \alpha, \beta + h^*, \delta t)$ distributed ($h^*$ is Esscher parameter) and $(e^{-rt}X_t)$ is a $\mathbb{Q}$ - martingale

• For each desired investment horizon $T$, we can construct an alternative pay-off with the same physical distribution as the CPPI distribution but at a lower cost
CPPI example cont’d

Path - independent pay-off generation algorithm:

1. Apply Monte Carlo methods to sample stock values under the $\mathbb{P}$ measure
2. For each i-th path keep track of the realizations $s_i$ and $v_i$, for the stock value $S_T$ and the portfolio value $V_T$. ($V_T = P_g$)
3. Realizations $v_i$ ($i = 1, \ldots, n$) are used to find the empirical distribution of the portfolio
4. By ordering the realizations $s_i$ and $v_i$ in increasing order and subsequently assigning each $v_i$ to the corresponding $s_i$ we find new path independent strategy that has the same empirical physical distribution. We keep the same physical distribution, whilst consistently reassigning the largest realizations for the pay-off for the largest values of the underlying stock
5. The cost is calculated with the realizations sampled under the risk neutral $\mathbb{Q}$ measure
Normal Inverse Guassian

| Inefficiency costs of CPPI strategy under $NIG(\mu, \alpha, \beta, \delta)$ |
|-------------------------------|------------------|
| multiplier | Time |
| 1 | 19 BP |
| 2 | 0.4 BP |
| 4 | 145 BP |
| 7 | 209 BP |

Geometric Brownian Motion

| Inefficiency costs of CPPI strategy under $N(\mu, \sigma)$ |
|-------------------------------|------------------|
| multiplier | Time |
| 1 | 1.25 BP |
| 2 | 0.07 BP |
| 4 | 12.8 BP |
| 7 | 48 BP |
Conclusions

- CPPI is a flexible, dynamic allocation technique with leverage and capital preservation, but

- CPPI is **Path-Dependent**

- We showed that while path dependent strategies gained commercial success they are value destroying, hence they are sub-optimal for investors with fixed investment horizon

- It is possible to create simple path-independent pay-off offering best trade-off between risk and return

- Complex path dependent strategies require expensive, dynamic hedging schemes whereas path independent pay-offs can be efficiently, statically hedged

- We showed that these results hold in Black-Scholes and Lévy markets
References


Questions?

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