1 The two-consumer pure exchange model, no production

1.1 Assumptions

There are two consumers 1 and 2. Each of them consume two types of goods, 1 and to to maximize utility. The utility function of the first consumer is a standard Cobb-Douglas function:

$$W_1 = A_1 x_{1,1}^{\alpha_1} x_{1,2}^{1-\alpha_1}$$

Similarly for the second consumer:

$$W_2 = A_2 x_{2,1}^{\alpha_2} x_{2,2}^{1-\alpha_2}$$

The Social Accounting matrix gives us the initial endowments of goods of both agents (how much they own before they can exchange goods with each other (the yellow part of the table) and how much they consume after they exchange (the blue part).

	X1	X2	CONS1	CONS2	
X1			40	70	110
X2			60	30	90
CONS1	50	50			100
CONS2	60	40			100
	110	90	100	100	

Value of consumers wealth
Consumption of goods

1.2 Solution of the model

Consumers maximize utility taking prices p_1 and p_2 as given. They initial wealths (incomes) are equal to the value of their endoments (denoted by all $w's$) at current prices:

$$INC_1 = w_{1,1}p_1 + w_{1,2}p_2 = 50p_1 + 50p_2$$

$$INC_2 = w_{2,1}p_1 + w_{2,2}p_2 = 60p_1 + 40p_2$$

Consumer's budget constraints are therefore:

$$p_1 x_{1,1} + p_2 x_{1,2} = w_{1,1}p_1 + w_{1,2}p_2$$

$$p_1 x_{2,1} + p_2 x_{2,2} = w_{2,1}p_1 + w_{2,2}p_2$$

The first consumer maximization problem is:

$$L_1 = A_1 x_{1,1}^{\alpha_1} x_{1,2}^{1-\alpha_1} - \lambda_1(p_1 x_{1,1} + p_2 x_{1,2} - w_{1,1}p_1 - w_{1,2}p_2)$$

and for the second consumer it will be symmetric:

$$L_2 = A_2 x_{2,1}^{\alpha_2} x_{2,2}^{1-\alpha_2} - \lambda_2(p_1 x_{2,1} + p_2 x_{2,2} - w_{2,1}p_1 - w_{2,2}p_2)$$

Once you take the first order conditions and solve them, the optimal consumption bundles will be:

$$\begin{array}{ll} x_{1,1} = \alpha_1 \frac{w_{1,1}p_1 + w_{1,2}p_2}{p_1}, & x_{2,1} = \alpha_2 \frac{w_{2,1}p_1 + w_{2,2}p_2}{p_1} \\ x_{1,2} = (1-\alpha_1)\frac{w_{1,1}p_1 + w_{1,2}p_2}{p_2}, & x_{2,2} = (1-\alpha_2)\frac{w_{2,1}p_1 + w_{2,2}p_2}{p_2} \end{array}$$

In the very simple version, them model could be formulated as two **market clearing equations (total endowment=total demand):**

$$w_{1,1} + w_{2,1} = \alpha_1 \frac{w_{1,1}p_1 + w_{1,2}p_2}{p_1} + \alpha_2 \frac{w_{2,1}p_1 + w_{2,2}p_2}{p_1}$$

$$w_{1,2} + w_{2,2} = (1-\alpha_1)\frac{w_{1,1}p_1 + w_{1,2}p_2}{p_2} + (1-\alpha_2)\frac{w_{2,1}p_1 + w_{2,2}p_2}{p_2}$$

In the above two equations, all $w's$ are exogeneous (given by the SAM) and all prices are endogeneous. Solution of the two equations gives us the ratio of prices. If we substitute prices equal 1 and 1, the demands $(x's)$ will be exactly as in the blue section of the SAM.

1.3 Models for computation

In our computation model we want to be able to analyze changes in utility and changes in price indices of aggregate consumptions of the two consumers. So first we need to define the price index of aggregate consumtion, pw_1 and pw_2. We will be treating one unit of aggregate consumption to be equal to one unit of utility. So we want to know what is minimum expenditure the consumers needs to make in order to buy one unit of utility. This is similar to a cost-minimization problem (this is again a Lagrange'an):

$$H_1 = p_1 x_{1,1} + p_2 x_{1,2} - \beta(A_1 x_{1,1}^{\alpha_1} x_{1,2}^{1-\alpha_1} - 1)$$

If take first order conditions with respect to $x_{1,1}$, $x_{1,2}$, and β , solve for $x's$ and put it back into $p_1 x_{1,1} + p_2 x_{1,2}$ part, you will see that:

$$pw_1 = p_1 x_{1,1}(W_1 = 1) + p_2 x_{1,2}(W_1 = 1) = \frac{1}{A_1}\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{p_2}{1-\alpha_1}\right)^{1-\alpha_1}$$

Formally we can call pw_1 a unit expenditure function (minimum expenditure to obtain one unit of utility).

Therefore:

$$pw_1 = \frac{1}{A_1}\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{p_2}{1-\alpha_1}\right)^{1-\alpha_1}$$

$$pw_2 = \frac{1}{A_2}\left(\frac{p_1}{\alpha_2}\right)^{\alpha_2}\left(\frac{p_2}{1-\alpha_2}\right)^{1-\alpha_2}$$

We also know that the budget constraint will be now:

$$pw_1 * W_1 = INC_1$$

$$pw_2 * W_2 = INC_2$$

Income is derived from endowments:

$$INC_1 = w_{1,1}p_1 + w_{1,2}p_2$$

$$INC_2 = w_{2,1}p_1 + w_{2,2}p_2$$

1.4 Final equations

Market clearing (note that I replaced incomes with the value of aggregate consumption):

$$w_{1,1} + w_{2,1} = \alpha_1 \frac{pw_1 * W_1}{p_1} + \alpha_2 \frac{pw_2 * W_2}{p_1}$$

$$w_{1,2} + w_{2,2} = (1-\alpha_1)\frac{pw_1 * W_1}{p_2} + (1-\alpha_2)\frac{pw_2 * W_2}{p_2}$$

Market clearing for aggregate consumption:

$$pw_1 * W_1 = INC_1$$

$$pw_2 * W_2 = INC_2$$

Price indices:

$$pw_1 = \frac{1}{A_1}\left(\frac{p_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{p_2}{1-\alpha_1}\right)^{1-\alpha_1}$$

$$pw_2 = \frac{1}{A_2}\left(\frac{p_1}{\alpha_2}\right)^{\alpha_2}\left(\frac{p_2}{1-\alpha_2}\right)^{1-\alpha_2}$$

Budget constraints:

$$INC_1 = w_{1,1}p_1 + w_{1,2}p_2$$

$$INC_2 = w_{2,1}p_1 + w_{2,2}p_2$$

1.5 Calibration

The *alpha*'s are equal to the shares of the particular good in agents total expenditure taken from SAM:

$$\alpha_1 = \frac{x_{1,1}^{SAM}}{x_{1,1}^{SAM} + x_{1,2}^{SAM}}$$

and others respectively.

A_1 and A_2 is chosen in such a way that aggregate utility is equal to aggregate consumption:

$$A_1 = \frac{W_1}{\left(x_{1,1}^{SAM}\right)^{\alpha_1} \left(x_{1,2}^{SAM}\right)^{1-\alpha_1}}$$

Endowments w are read from yellow part of the SAM.