CGE models in linearized form

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Model in non-linear form:

Consumer

Budget constraint

$$PW \cdot W = INC \tag{1}$$

Consumer price index

$$PW = \frac{1}{A}\left[\frac{P_X}{\alpha_X}\right]^{\alpha_X}\left[\frac{P_Y}{\alpha_Y}\right]^{\alpha_Y} \tag{2}$$

Income of consumers

$$INC = P_K\omega_K + P_L\omega_L \tag{3}$$

Producer and market clearing

Zero profit condition ($P = MC$, for $j = X, Y$) for $Y = 1$:

$$P_j = \frac{1}{B_j}\left[\frac{P_L}{\beta_{L,j}}\right]^{\beta_{L,j}}\left[\frac{P_K}{\beta_{K,j}}\right]^{\beta_{K,j}} \tag{4}$$

Market clearing (for factor $i = K, L$):

$$\omega_i = \beta_{i,X}\frac{P_X XS}{P_i} + \beta_{i,Y}\frac{P_Y YS}{P_i} \tag{5}$$

Market clearing for goods.

$$XS = \alpha_X \frac{PW \cdot W}{P_X}, \;\; YS = \alpha_Y \frac{PW \cdot W}{P_Y} \tag{6}$$

Solution of 1-6 ensures model equilibrium.

Linearization: budget constraint

- capital letters: levels, lower case letters: percentage changes:
- $$p = \frac{\Delta P}{P}$$
- Start linearization from 1:

$$pw + w = inc$$

- How do we get this? Lest totally differentiate:

$$dPW \cdot W + dW \cdot PW = dINC$$

Linearization

Divide both sides by INC :

$$dP \cdot W/INC + dW \cdot P/INC = dINC/INC$$

Substitute: $INC = PW \cdot W$

$$\frac{dPW \cdot W}{PW \cdot W} + \frac{dW \cdot PW}{PW \cdot W} = inc$$

$$pw + w = inc \tag{7}$$

Linearization: price index

Linearization of 2 is a bit more difficult:

$$dPW = \alpha_X \frac{1}{A}\left[\frac{1}{\alpha_X}\right]^{\alpha_X} P_X^{\alpha_X - 1}\left[\frac{P_Y}{\alpha_Y}\right]^{\alpha_Y} dP_X + \alpha_Y \frac{1}{A}\left[\frac{1}{\alpha_Y}\right]^{\alpha_Y} P_Y^{\alpha_Y - 1}\left[\frac{P_X}{\alpha_X}\right]^{\alpha_X} dP_Y$$

Divide by PW:

$$pw = \frac{\alpha_X \frac{1}{A}\left[\frac{1}{\alpha_X}\right]^{\alpha_X} P_X^{\alpha_X - 1}\left[\frac{P_Y}{\alpha_Y}\right]^{\alpha_Y} dP_X}{\frac{1}{A}\left[\frac{P_X}{\alpha_X}\right]^{\alpha_X}\left[\frac{P_Y}{\alpha_Y}\right]^{\alpha_Y}} + \frac{\alpha_Y \frac{1}{A}\left[\frac{1}{\alpha_Y}\right]^{\alpha_Y} P_Y^{\alpha_Y - 1}\left[\frac{P_X}{\alpha_X}\right]^{\alpha_X} dP_Y}{\frac{1}{A}\left[\frac{P_X}{\alpha_X}\right]^{\alpha_X}\left[\frac{P_Y}{\alpha_Y}\right]^{\alpha_Y}}$$

$$pw = \frac{\alpha_X dP_X}{P_X} + \frac{\alpha_Y dP_Y}{P_Y}$$

which gives:

$$pw = \alpha_X p_X + \alpha_Y p_Y \tag{8}$$

Linearization: income

Eqaution 3:

$$dINC = \omega_K dP_K + P_K d\omega_K + P_L d\omega_L + \omega_L dP_L$$

Divide by INC :

$$inc = \frac{\omega_K dP_K}{INC} + \frac{P_K d\omega_K}{INC} + \frac{P_L d\omega_L}{INC} + \frac{\omega_L dP_L}{INC}$$

$$inc = \frac{P_K \cdot \omega_K dP_K}{INC \cdot P_K} + \frac{\omega_K \cdot P_K d\omega_K}{INC \cdot \omega_K} + \frac{\omega_L \cdot P_L d\omega_L}{INC \cdot \omega_L} + \frac{P_L \cdot \omega_L dP_L}{INC \cdot P_L}$$

So ($sc_{INC,K}$ oraz $sc_{INC,L}$) are the factor shares in income:

$$inc = (p_K + \frac{d\omega_K}{\omega_K}) \cdot sc_{INC,K} + (p_L + \frac{d\omega_L}{\omega_L}) \cdot sc_{INC,L} \qquad (9)$$

Linearization: zero profit

$$P_j = \frac{1}{B_j}\left[\frac{P_L}{\beta_{L,j}}\right]^{\beta_{L,j}}\left[\frac{P_K}{\beta_{K,j}}\right]^{\beta_{K,j}}$$

We can rewrite as

$$p_j = \beta_{L,j} p_L + \beta_{K,j} p_K \tag{10}$$

Linearization: market clearing

$$XS = \alpha_X \frac{PW \cdot W}{P_X}, \;\; YS = \alpha_Y \frac{PW \cdot W}{P_Y}$$

$$dXS = \alpha_X \left(\frac{PW}{P_X} dW + \frac{W}{P_X} dPW - \frac{PW \cdot W}{P_X} \cdot \frac{dPX}{P_X} \right) \| : XS$$

$$xs = w + pw - p_X \;\; ys = w + pw - p_Y \tag{11}$$

We see the price elasticity of demand equal to one.

Linearization: factor market

We linearize equation

$$\omega_i = \beta_{i,X} \frac{P_X XS}{P_i} + \beta_{i,Y} \frac{P_Y YS}{P_i}$$

So: (for factor $i = K, L$):

$$d\omega_i = \beta_{i,X}\left(\frac{P_X dXS}{P_i} + \frac{dP_X XS}{P_i} - \frac{P_X XS}{P_i}\frac{dP_i}{P_i}\right) + \beta_{i,Y}\left(\frac{P_Y dYS}{P_i} + \frac{dP_Y YS}{P_i} - \frac{P_Y YS}{P_i}\frac{dP_i}{P_i}\right)$$

$$d\omega_i = \beta_{i,X}\left(P_X dXS + dP_X XS - P_X XS \frac{dP_i}{P_i}\right) + \beta_{i,Y}\left(P_Y dYS + dP_X YS - P_X XS \frac{dP_i}{P_i}\right)$$

$$d\omega_i / w_i = sw_{i,X}(xs + p_s) + sw_{i,Y}(xs + p_s) - p_i \tag{12}$$

where: $sw_{i,X} = \frac{\beta_{i,X} P_X XS}{\beta_{i,X} P_X XS + \beta_{i,Y} P_Y YS}$ and $sw_{i,Y} = \frac{\beta_{i,Y} P_Y YS}{\beta_{i,X} P_X XS + \beta_{i,Y} P_Y YS}$.

The linearized model

$$pw + w = inc \tag{13}$$

$$pw = \alpha_X p_X + \alpha_Y p_Y \tag{14}$$

$$inc = (p_K + \frac{d\omega_K}{\omega_K}) \cdot sc_{INC,K} + (p_L + \frac{d\omega_L}{\omega_L}) \cdot sc_{INC,L} \tag{15}$$

$$p_j = \beta_{L,j} p_L + \beta_{K,j} p_K \tag{16}$$

$$xs = w + pw - p_X \;\; ys = w + pw - p_Y \tag{17}$$

$$d\omega_i / w_i = sw_{i,X}(xs + p_s) + sw_{i,Y}(xs + p_s) - p_i \tag{18}$$

The linearized model

- The Cobb-Douglas shares will be calibrated as before
- We still have to compute from SAM: $sw_{i,X} = \frac{\beta_{i,X} P_X XS}{\beta_{i,X} P_X XS + \beta_{i,Y} P_Y YS}$ and $sw_{i,Y} = \frac{\beta_{i,Y} P_Y YS}{\beta_{i,X} P_X XS + \beta_{i,Y} P_Y YS}$.
- $sc_{INC,K} = \frac{P_K \omega_K}{INC}$ and $sc_{INC,L} = \frac{P_L \omega_L}{INC}$
- In our model all variables in the benchmark run will be equal to zero
- The reactions to shocks will be all expressed in percentage changes (multiply by initial levels to recover post-simulation levels).