Example small open production economy with Cobb-Douglas function

Consider an economy producing two goods: 1 and 2 using two factors of production: L and K. The factor prices will be denoted by w_L and w_K. The prices of final goods are exogeneous (small open economy) and equal to p_1 and p_2. The firms producing goods 1 and 2 can sell any amount of goods at these prices. The total endoments of factors L and K are equal to ω_L and ω_K. Technologies are: $q_1 = z_{L1}^{\alpha_1} z_{K1}^{1-\alpha_1}$ and $q_2 = z_{L2}^{\alpha_1} z_{K2}^{1-\alpha_2}$. We assume that $\alpha_1 > \alpha_2$.

Step 1: determine the cost functions and factor demads of the two firms.

How? for both firms do the cost minimization - fortunately we know the results already. The problem is (for firm 1, we have a similar one for firm 2):

$$\mathcal{L}_1 = w_L z_{L1} + w_K z_{K1} - \lambda(z_{L1}^{\alpha_1} z_{K1}^{1-\alpha_1} - q_1) \tag{1}$$

and the solution to the problem is:

$$z_{L1} = \alpha_1 \frac{C_1(w_L, w_K)}{w_L}, \tag{2}$$

$$z_{K1} = (1-\alpha_1) \frac{C_1(w_L, w_K)}{w_K}, \tag{3}$$

where

$$C_1(w_L, w_K, q_1) = \alpha_1^{-\alpha_1}(1-\alpha_1)^{-(1-\alpha_1)} w_L^{\alpha_1} w_K^{1-\alpha_1} q_1. \tag{4}$$

And by analogy:

$$z_{L2} = \alpha_2 \frac{C_2(w_L, w_K)}{w_L}, \tag{5}$$

$$z_{K2} = (1-\alpha_2) \frac{C_2(w_L, w_K)}{w_K}, \tag{6}$$

where

$$C_2(w_L, w_K, q_2) = \alpha_2^{-\alpha_2}(1-\alpha_2)^{-(1-\alpha_2)} w_L^{\alpha_2} w_K^{1-\alpha_2} q_2. \tag{7}$$

Step 2: Profit maximization.

Firms maximize profits:

$$\pi_j = p_j q_j - C_j(w_L, w_K, q_j) \tag{8}$$

First order condition for all j :

$$p_j = c_j(w_L, w_K) \tag{9}$$

Where $c_j(w_L, w_K) = \frac{\partial C_j(w_L, w_K, q_j)}{\partial q_j}$ is the marginal cost = average cost = unit cost (CRS) which in this case is independent of q_j.
So we have:

$$p_1 = \alpha_1^{-\alpha_1}(1-\alpha_1)^{-(1-\alpha_1)} w_L^{\alpha_1} w_K^{1-\alpha_1}$$

$$p_2 = \alpha_2^{-\alpha_2}(1-\alpha_2)^{-(1-\alpha_2)} w_L^{\alpha_2} w_K^{1-\alpha_2}$$

or:

$$p_1 = A_1 w_L^{\alpha_1} w_K^{1-\alpha_1}$$

$$p_2 = A_2 w_L^{\alpha_2} w_K^{1-\alpha_2}$$

Therefore we can solve the above system of equations:

$$w_K = p_2^{\frac{1}{1-\alpha_2}} A_2^{\frac{-1}{1-\alpha_2}} w_L^{\frac{-\alpha_2}{1-\alpha_2}}$$

and:

$$p_1 = A_1 w_L^{\alpha_1} p_2^{\frac{1-\alpha_1}{1-\alpha_2}} A_2^{\frac{\alpha_1-1}{1-\alpha_2}} w_L^{\frac{-\alpha_2(1-\alpha_1)}{1-\alpha_2}}$$

$$p_1 = A_1 w_L^{\alpha_1} p_2^{\frac{1-\alpha_1}{1-\alpha_2}} A_2^{\frac{\alpha_1-1}{1-\alpha_2}} w_L^{\frac{\alpha_1(1-\alpha_2)-\alpha_2(1-\alpha_1)}{1-\alpha_2}}$$

$$p_1 p_2^{-\frac{1-\alpha_1}{1-\alpha_2}} A_1^{-1} A_2^{\frac{1-\alpha_1}{1-\alpha_2}} = w_L^{\frac{\alpha_1-\alpha_2}{1-\alpha_2}}$$

$$p_1^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}} p_2^{-\frac{1-\alpha_1}{\alpha_1-\alpha_2}} A_1^{-\frac{1-\alpha_2}{\alpha_1-\alpha_2}} A_2^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}} = w_L$$

$$p_1^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}} p_2^{-\frac{1-\alpha_1}{\alpha_1-\alpha_2}} A_1^{-\frac{1-\alpha_2}{\alpha_1-\alpha_2}} A_2^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}} = w_L$$

$$w_L = p_1^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}} p_2^{-\frac{1-\alpha_1}{\alpha_1-\alpha_2}} A_1^{-\frac{1-\alpha_2}{\alpha_1-\alpha_2}} A_2^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}}$$

$$w_K = p_2^{\frac{1}{1-\alpha_2}} A_2^{\frac{-1}{1-\alpha_2}} \left(p_1^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}} p_2^{-\frac{1-\alpha_1}{\alpha_1-\alpha_2}} A_1^{-\frac{1-\alpha_2}{\alpha_1-\alpha_2}} A_2^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}} \right)^{\frac{-\alpha_2}{1-\alpha_2}}$$

$$w_K = p_1^{\frac{-\alpha_2}{\alpha_1-\alpha_2}} p_2^{\frac{\alpha_1}{\alpha_1-\alpha_2}} A_1^{\frac{\alpha_2}{\alpha_1-\alpha_2}} A_2^{\frac{-\alpha_1}{\alpha_1-\alpha_2}}$$

To make things even simpler, we can introduce two new constants, $A_L = A_1^{-\frac{1-\alpha_2}{\alpha_1-\alpha_2}} A_2^{\frac{1-\alpha_1}{\alpha_1-\alpha_2}}$ and $A_K = A_1^{\frac{\alpha_2}{\alpha_1-\alpha_2}} A_2^{\frac{-\alpha_1}{\alpha_1-\alpha_2}}$:

$$w_L = p_1^{\frac{1-\alpha_2}{\alpha_1-\alpha_2}} p_2^{-\frac{1-\alpha_1}{\alpha_1-\alpha_2}} A_L$$

$$w_K = p_1^{\frac{-\alpha_2}{\alpha_1-\alpha_2}} p_2^{\frac{\alpha_1}{\alpha_1-\alpha_2}} A_K$$

So if the prices are known, we know exactly what the wages will be.
Note that an increase in p_1 causes an increase in w_L and a decrease in w_K if $\alpha_1 > \alpha_2$. (Stolper Samuelson).

Step 3: Factor demands

Note that with constant returns to scale:

$$C_j(w_L, w_K, q_j) = c_j(w_L, w_K) q_j$$

where c_j is marginal cost of production equal to: $c_j(w_L, w_K) = A_j w_L^{\alpha_1} w_K^{1-\alpha_1}$.
Therefore, our factor demands can be expressed as:

$$z_{Lj} = \alpha_i \frac{c_j(w_L, w_K)}{w_L} q_j$$

$$z_{Kj} = (1-\alpha_i)\frac{c_j(w_L, w_K)}{w_K}q_j$$

we can also call the $a_{Lj} = a_{Lj}(w_L, w_K) = \alpha_j \frac{c_j(w_L,w_K)}{w_L}$ and $a_{Kj} = a_{Kj}(w_L, w_K) = (1-\alpha_j)\frac{c_j(w_L,w_K)}{w_L}$, the **unit factor requirement, so that:**

$$z_{ij} = a_{ij}q_j$$

Note that: a_{ij} is constant:

- as long as wages are constant
- wages are constant as long as prices are constant.

Note also that for any w_L and w_K, $z_{L1} > z_{L2}$.

Step 4: Resource constraint

(the factor use in both sectors have to be equal to respective endowments):

$$z_{L1}(w_L, w_K, q_1) + z_{L2}(w_L, w_K, q_2) = \omega_L$$

$$z_{K1}(w_L, w_K, q_1) + z_{K2}(w_L, w_K, q_2) = \omega_K$$

Making use of the a_{ij} (remember that we know it ex-ante with exogeneous prices $p_1$and p_2).

$$q_1 a_{L1} + q_2 a_{L2} = \omega_L$$

$$q_1 a_{K1} + q_2 a_{K2} = \omega_K$$

This is a system of two linear equations with unknowns q_1 and q_2. **Solution to the system gives us output levels.** Let us see:

$$q_2 a_{K2} = \omega_K - q_1 a_{K1}$$

$$q_2 = \frac{\omega_K - q_1 a_{K1}}{a_{K2}}$$

$$q_1 a_{L1} + \frac{\omega_K - q_1 a_{K1}}{a_{K2}} a_{L2} = \omega_L$$

$$q_1 \left(a_{L1} - \frac{a_{K1}a_{L2}}{a_{K2}} \right) = \omega_L - \frac{\omega_K a_{L2}}{a_{K2}}$$

$$q_1 \left(\frac{a_{L1}a_{K2} - a_{K1}a_{L2}}{a_{K2}} \right) = \omega_L - \frac{\omega_K a_{L2}}{a_{K2}}$$

$$q_1 = \frac{1}{a_{L1}a_{K2} - a_{K1}a_{L2}}(a_{K2}\omega_L - a_{L2}\omega_K)$$

$$q_2 = \frac{\omega_K}{a_{K2}} - \frac{a_{K1}}{a_{K2}} \frac{1}{a_{L1}a_{K2} - a_{K1}a_{L2}}(a_{K2}\omega_L - \omega_K a_{L2})$$

$$q_2 = \frac{\omega_K a_{L1}a_{K2} - a_{K1}a_{K2}\omega_L}{a_{K2}(a_{L1}a_{K2} - a_{K1}a_{L2})}$$

$$q_1 = \frac{1}{a_{L1}a_{K2} - a_{K1}a_{L2}}(a_{K2}\omega_L - a_{L2}\omega_K)$$

$$q_2 = \frac{1}{a_{L1}a_{K2} - a_{K1}a_{L2}}(a_{L1}\omega_K - a_{K1}\omega_L)$$

Since due to our assumptions: $a_{L1}a_{K2} > a_{K1}a_{L2}$, an increase in ω_L will result in an increse in q_1 and a decrease in q_2. An increase in ω_K will result in an increse in q_2 and a decrease in q_1. (Rybczynski)

Step 5: Get the factor demands.

Substitute $a'_{ij}s$ and q'_js into the factor demands and obtain the factor demand $z'_{ij}s$.