

More flexible utility and production function. Constant elasticity of substitution

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CES utility function

In the two good case:

$$U = \theta(a_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + a_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$$

In the N -good case:

$$U = \theta\left(\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

Deriving Marshallian demand

Consumer maximizes utility subject to the budget constraint:

$$I = \sum_{i=1}^N p_i x_i$$

The Lagrange function:

$$L = \left(\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} - \lambda(I - \sum_{i=1}^N p_i x_i)$$

The first order conditions:

$$\frac{\partial L}{\partial x_i} = \frac{\sigma}{\sigma-1} \theta \left(\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} \frac{\sigma-1}{\sigma} a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}-1} - \lambda p_i = 0$$

$$\frac{\partial L}{\partial \lambda} = I - \sum_{i=1}^N p_i x_i = 0$$

Simplify the first FOC:

$$\theta \left(\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} a_i^{\frac{1}{\sigma}} x_i^{\frac{-1}{\sigma}} = \lambda p_i$$

Write down the same thing for j 'th good:

$$\theta \left(\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} a_j^{\frac{1}{\sigma}} x_j^{\frac{-1}{\sigma}} = \lambda p_j$$

Divide one by another:

$$\frac{a_i^{\frac{1}{\sigma}} x_i^{\frac{-1}{\sigma}}}{a_j^{\frac{1}{\sigma}} x_j^{\frac{-1}{\sigma}}} = \frac{p_i}{p_j}$$

$$\frac{a_i^{\frac{1}{\sigma}} x_j^{\frac{1}{\sigma}}}{a_j^{\frac{1}{\sigma}} x_i^{\frac{1}{\sigma}}} = \frac{p_i}{p_j}$$

$$x_j = \left(\frac{p_i}{p_j} \right)^{\sigma} \frac{a_j x_i}{a_i}$$

Now plug it in the (slightly rewritten) budget constraint:

$$I = \sum_{j=1}^N p_j x_j = \sum_{j=1}^N p_j \left(\frac{p_i}{p_j} \right)^{\sigma} \frac{a_j x_i}{a_i} = \frac{x_i}{a_i} p_i^{\sigma} \sum_{j=1}^N p_j^{1-\sigma}$$

$$I = \frac{x_i}{a_i} p_i^{\sigma} \sum_{j=1}^N p_j^{1-\sigma}$$

$$x_i = a_i I \frac{\sum_{j=1}^N p_j^{1-\sigma}}{p_i^{\sigma}}$$

Deriving Hicksian demand

We will work with nested functions, so Hicksian demand would be more suitable. The consumer problem:

$$L = \sum_{i=1}^N p_i x_i - \lambda (\theta (\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} - U)$$

The first order conditions:

$$\frac{\partial L}{\partial x_i} = p_i - \lambda \frac{\sigma}{\sigma-1} \theta (\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}-1} \frac{\sigma-1}{\sigma} a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = \theta (\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} - U = 0$$

Rearranging the first condition:

$$p_i = \lambda \theta (\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} a_i^{\frac{1}{\sigma}} x_i^{\frac{-1}{\sigma}}$$

Take the same for j th good:

$$p_j = \lambda \theta (\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} a_j^{\frac{1}{\sigma}} x_j^{\frac{-1}{\sigma}}$$

Take the ratio of the two:

$$\frac{a_i^{\frac{1}{\sigma}} x_i^{\frac{-1}{\sigma}}}{a_j^{\frac{1}{\sigma}} x_j^{\frac{-1}{\sigma}}} = \frac{p_i}{p_j}$$

$$\frac{a_i^{\frac{1}{\sigma}} x_j^{\frac{1}{\sigma}}}{a_j^{\frac{1}{\sigma}} x_i^{\frac{1}{\sigma}}} = \frac{p_i}{p_j}$$

$$x_j = \left(\frac{p_i}{p_j} \right)^{\sigma} \frac{a_j x_i}{a_i}$$

Plug it now into the utility function (second constraint):

$$\theta \left(\sum_{j=1}^N a_j^{\frac{1}{\sigma}} x_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = U$$

$$\theta \left(\sum_{j=1}^N a_j^{\frac{1}{\sigma}} \left[\left(\frac{p_i}{p_j} \right)^{\sigma} \frac{a_j x_i}{a_i} \right]^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = U$$

Now the hell of different powers:

$$U = p_i^{\sigma} \frac{x_i}{a_i} \theta \left(\sum_{j=1}^N a_j^{\frac{1}{\sigma}} p_j^{1-\sigma} a_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

$$\frac{U a_i (\sum_{j=1}^N a_j p_j^{1-\sigma})^{\frac{\sigma}{1-\sigma}}}{\theta p_i^{\sigma}} = x_i$$

Define P - a unit CES expenditure function (a CES price index):

$$P = \frac{1}{\theta} \left(\sum_{j=1}^N a_j p_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

then:

$$x_i = \theta^{\sigma-1} \frac{a_i}{p_i^{\sigma}} P^{\sigma} U$$

is the demand for good i given the level of utility and the price index of consumption.

Verify the **expenditure function**:

$$E = \sum_i p_i x_i = \sum_i p_i \theta^{\sigma-1} \frac{a_i}{p_i^{\sigma}} P^{\sigma} U = U \frac{1}{\theta^{\sigma}} \left(\sum_{j=1}^N a_j p_j^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} \sum_i a_i p_i^{1-\sigma} \theta^{\sigma-1} =$$

$$U \left(\sum_{j=1}^N a_j p_j^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} \sum_i a_i p_i^{1-\sigma} \theta^{-1} = U \frac{1}{\theta} \left(\sum_{j=1}^N a_j p_j^{1-\sigma} \right)^{\frac{1}{1-\sigma}} = U P$$

So the price index P is the unit expenditure function (for $U = 1$).

Calibration (this is a little nasty with CES)

- set σ to your external estimate (get it using econometric methods or obtain for literature or guess)
- what about other parameters:

$$\frac{x_i p_i^\sigma}{P^\sigma U} \theta^{1-\sigma} = a_i$$

$$\frac{p_i x_i^{\frac{1}{\sigma}}}{P U^{\frac{1}{\sigma}}} \theta^{\frac{1-\sigma}{\sigma}} = a_i^{\frac{1}{\sigma}}$$

$$\frac{p_i x_i x_i^{\frac{\sigma-1}{\sigma}}}{P U U^{\frac{\sigma-1}{\sigma}}} \theta^{\frac{1-\sigma}{\sigma}} = a_i^{\frac{1}{\sigma}}$$

$$S_i \frac{x_i^{\frac{\sigma-1}{\sigma}}}{U^{\frac{\sigma-1}{\sigma}}} \theta^{\frac{1-\sigma}{\sigma}} = a_i^{\frac{1}{\sigma}}$$

$$\sum_i S_i \left(\frac{x_i}{U}\right)^{\frac{\sigma-1}{\sigma}} \theta^{\frac{1-\sigma}{\sigma}} = 1$$

where $S_i = \frac{p_i x_i}{P U}$ is the share in expenditure. Therefore:

$$\theta = \left[\sum_i S_i \left(\frac{x_i}{U}\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

$$S_i \frac{x_i^{\frac{\sigma-1}{\sigma}}}{U^{\frac{\sigma-1}{\sigma}}} \theta^{\frac{1-\sigma}{\sigma}} = a_i^{\frac{1}{\sigma}}$$

$$\frac{p_i x_i x_i^{\frac{1-\sigma}{\sigma}}}{\sum_i p_i x_i x_i^{\frac{1-\sigma}{\sigma}}} = a_i^{\frac{1}{\sigma}}$$

$$\frac{p_i x_i^{\frac{1}{\sigma}}}{\sum_i p_i x_i^{\frac{1}{\sigma}}} = a_i^{\frac{1}{\sigma}}$$

Summing up, to calibrate:

- once σ is chosen, set a_i to $a_i = \left(\frac{p_i x_i^{\frac{1}{\sigma}}}{\sum_i p_i x_i^{\frac{1}{\sigma}}} \right)^\sigma$
- and consequently: $\theta = U / (\sum_{i=1}^N a_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$.

CES production function

By analogy with the Hicksian demand we can state the CES production function:

$$Q = A \left(\sum_{i=1}^K b_i^{\frac{1}{\gamma}} v_i^{\frac{\gamma-1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}},$$

where A , b_i and γ are parameters, Q is the output level of the produced good and v_i is the amount of the i -th input.

The demand for inputs is then:

$$v_i = A^{\gamma-1} \frac{b_i}{w_i^{\gamma}} p^{\gamma} Q,$$

where w_i is the wage of the i -th input and p is the price of the final output (since price equals marginal cost). The marginal (unit) cost of production has to equal the price of output:

$$p_i = \frac{1}{A} \left(\sum_{i=1}^K b_i w_i^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

Calibration

- set γ to your external estimate (get it using econometric methods or obtain for literature or guess)
- $b_i = \left(\frac{w_i v_i^{\frac{1}{\gamma}}}{\sum_i w_i v_i^{\frac{1}{\gamma}}} \right)^{\gamma}$
- With $w_i = 1$ or PFAC=1 in the initial equilibrium, we can write the parameter value as:

```
B(FAC)=
(USE0(FAC,SEC)**(1 / GAMMA(SEC))
/ SUM(FACC, USE0(FACC,SEC)** (1 / GAMMA(SEC)) ))**GAMMA(SEC) ;
```

- $A = Q / (\sum_{i=1}^K b_i^{\frac{1}{\gamma}} v_i^{\frac{\gamma-1}{\gamma}})^{\frac{\gamma}{\gamma-1}}$, which in your code will look like:

```
ACES(SEC) = XD0(SEC)
/(SUM(FAC,
      B(FAC,SEC)**(1 / GAMMA(SEC)) * USE0(FAC,SEC)**((GAMMA(SEC)-1) / GAMMA(SEC)))
```