The plan

- Inputs and outputs
- Production set
- Transformation vs production function
- Examples
The setup

- We will be talking about production processes.
- Production process: make commodities using other commodities.
- Definitions
  - $L$ commodities/goods
  - Commodities can be inputs or outputs
  - Production plan (vector):
    \[ y = (y_1, \ldots, y_L) \in \mathbb{R}^L \]
  - Production set $Y \in \mathbb{R}^L$
    - Set of all $y$'s that are possible or *feasible* under current technology and inputs.
Note that in the most general setup each good can be an input or an output.

- input $< 0$, output $> 0$

**Example 5.B.1, $L = 5$, $y = (-5, 2, -6, 3, 0)$**

- in the production plan $y$, goods 1 and 3 are inputs (used in the quantity of 5 and 6),
- goods 2 and 3 are outputs.
- Good 5 is not produced nor its an input.
Transformation function

given the production set $Y$ we can define a *transformation* function.
it defines the set $Y$:

$$Y = \{ y \in \mathbb{R}^L : F(y) \leq 0 \}$$

All points where $F(y) = 0$ are on the boundary of $y$. \(\rightarrow\) **transformation frontier**.
For example: $-\sqrt{-y_1} + y_2 \leq 0$. $y_1$ is an input and $y_2$ is an output.
But we could have: $-\sqrt{-y_1} + y_2 + y_3 \leq 0$. Two outputs and one input.
Transformation with one input and one output

\[ Y = \{ y : F(y) \leq 0 \} \]

\[ \text{Slope} = -MRT_{12}(\bar{y}) \]

Transformation frontier
\[ \{ y : F(y) = 0 \} \]
If $F(\cdot)$ is differentiable, and we are at the frontier ($F(\bar{y}) = 0$) then we define marginal rate of transformation as:

$$MRT_{lk}(\bar{y}) = \frac{\partial F(\bar{y})}{\partial l} \cdot \frac{\partial F(\bar{y})}{\partial k}$$

for each $l$ and $k$.

What is it? Totally differentiate $F(\bar{y}) = 0$. For $dl \neq 0$ and $dk \neq 0$ and all other $d\ldots$’s equal to zero:

$$\frac{\partial F(\cdot)}{\partial l} dl + \frac{\partial F(\cdot)}{\partial k} dk = 0$$
Marginal rate of transformation

\[
\frac{\partial F(\cdot)}{\partial l} dl + \frac{\partial F(\cdot)}{\partial k} dk = 0
\]

rearranging gives:

\[
\frac{\partial F(\cdot)}{\partial l} = -\frac{dk}{dl} = MRT_{lk}(\bar{y})
\]

So:

- if both goods are outputs, it shows the tradeoff between production of goods
- if one of goods is an input and other is output, it shows the marginal productivity of an input in terms of an output
- if both goods are inputs, it shows Marginal Rate of Technical Substitution (MRTS) - more on this later.
Properties of production sets

- $Y$ is nonempty
- $Y$ is closed (includes its boundary)
- Other properties:
  - No free lunch
  - Possibility of inaction $0 \in Y$ (no sunk costs)
  - Free disposal (extra amounts of inputs can be eliminated at no costs).
  - Irreversibility (you cannot produce inputs from outputs)
  - Non-increasing returns to scale
  - Non-decreasing returns to scale
  - Constant returns to scale.
  - Convexity
Violation of no free lunch (left)
Sunk costs

![Graph showing sunk costs](image-url)
Returns to scale

- Nonincreasing returns to scale: $\alpha y \in Y, \alpha < 1$ (convexity - left - any production level can be scaled down)
- Nondecreasing returns to scale: $\alpha y \in Y, \alpha > 1$ (any production level can be scaled up)
Nonincreasing returns to scale: $\alpha y \in Y$, $\alpha > 0$ (convexity - left)
Let \( q = (q_1, \ldots, q_M) \geq 0 \) be outputs and \( z = (z_1, \ldots, z_{L-M}) \geq 0 \) be inputs.

In that case we measure inputs and outputs in nonnegative numbers (unlike before)

A commonly use example: one output, many inputs

- We can define a **production function** \( f(\cdot) \) that gives rise to the production set

\[
Y = \{(-z_1, \ldots, -z_{L-1}, q) : q - f(z_1, \ldots, z_{L-1}) \leq 0\} \text{ and } (z_1, \ldots, z_{L-1}) \geq 0
\]

- If we are on boundary: \( q = f(z_1, \ldots, z_{L-1}) \)
Given the production function $f(z_1, \ldots, z_{L-1})$, we define the marginal rate of technical substitution for input $l$ and $k$ as:

$$MRTS_{lk}(\bar{z}) = \frac{\partial f(\bar{z})}{\partial l} \frac{\partial l}{\partial f(\bar{z})} \frac{\partial f(\bar{z})}{\partial k}$$

When we start at the level of production $q = f(\bar{z})$ and the initial allocation of inputs $\bar{z}$, then it measures how much we have to add of one input if we take away some of the other

→ analogy with the consumer theory and $MRS$. 

Jan Hagemejer
Advanced Microeconomics
Returns to scale

- constant: $f(\alpha z) = \alpha f(z)$ for any $\alpha > 0$
- increasing: $f(\alpha z) > \alpha f(z)$ for any $\alpha > 1$
- decreasing: $f(\alpha z) < \alpha f(z)$ for any $\alpha > 1$
Suppose we have just one output $q$:

Isoquant:

$$Q(y) = \{ z \in \mathbb{R}_L^+ : (q, -z) \in Y \}$$

so it shows all combination of inputs that produce a given output $q$.

Input requirement set - a set of inputs that produce at least the output $q$. (area above isoquant).
with two inputs and one output. You can *substitute* inputs $z_1$ and $z_2$ in combinations given by $a_1$ and $a_2$ (a’s are marginal productivities)

$$q = f(z) = a_1 z_1 + a_2 z_2.$$ 

Equation of the isoquant at $q$: $q/a_2 - a_1/a_2 z_1 = z_2$.

How does it look like?
What is the MRTS?
Cobb-Douglas

2 inputs, one output:

\[ q = f(z) = z_1^\alpha z_2^{1-\alpha} \]

\[ Y = \{(q, -z) \in \mathbb{R}^3 : q \leq z_1^\alpha z_2^{1-\alpha}\} \]

\[ Q(q, -z) = \{(z) \in \mathbb{R}_+^2 : q = z_1^\alpha z_2^{1-\alpha}\} \]
2 inputs, one output:

\[ q = f(z) = \min(a_1 z_1, a_2 z_2) \]

\[ \frac{1}{a_i} \] is the unit factor requirement - minimal input for one unit of output

\[ Y = \{(q, -z) \in \mathbb{R}^3 : q \leq \min(a_1 z_1, a_2 z_2)\} \]

\[ Q(q, -z) = \{(q, -z) \in \mathbb{R}_+^2 : q = \min(a_1 z_1, a_2 z_2)\} \]
Elasticity of substitution

- percentage change in the MRTS with a change in the ratio of inputs given fixed output

or: the curvature of isoquant (how the ratio of factor inputs changes as the slope of the isoquant changes).

\[ \sigma = \frac{\Delta (z_2/z_1)}{z_2/z_1} \frac{\Delta MRTS}{MRTS} \]

or in derivative terms: \( \sigma = \frac{MRTS}{z_2/z_1} \frac{d(z_2/z_1)}{dMRTS} \).

or approximation in logs: \( \sigma = \frac{d \ln (z_2/z_1)}{d \ln |MRTS|} \).

- For Cobb-Douglas \( \sigma = 1 \) - it is (constant elasticity of substitution function)
A more general CES

(2 inputs)

\[ q = \left( a_1 z_1^\rho + a_2 z_2^\rho \right)^{\frac{1}{\rho}} \]

- constant returns to scale
- \( \sigma = \frac{1}{1-\rho} \) is the elasticity of substitution
- in particular, as \( \rho \to 0 \) the properties of CES tend to Cobb-Douglas