Advanced Microeconomics

Production

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The plan

- Inputs and outputs
- Production set
- Transformation vs production function
- Examples

The setup

- We will be talking about production processes
- Production process: make commodities using other commodities
- Definitions
  - L commodities/goods
  - commodities can be inputs or outputs
  - production plan (vector):

$$y = (y_1, \dots, y_L) \in \mathbb{R}^L$$

  - production set $Y \in \mathbb{R}^L$
    - set of all y's that are possible or *feasible* under current: technology and inputs

Example

- Note that in the most general setup each good can be an input or an output.
    - input < 0, output > 0
- Example 5.B.1, $L = 5$, $y = (-5, 2, -6, 3, 0)$
    - in the production plan y, goods 1 and 3 are inputs (used in the quantity of 5 and 6),
    - goods 2 and 3 are outputs.
    - Good 5 is not produced nor its an input

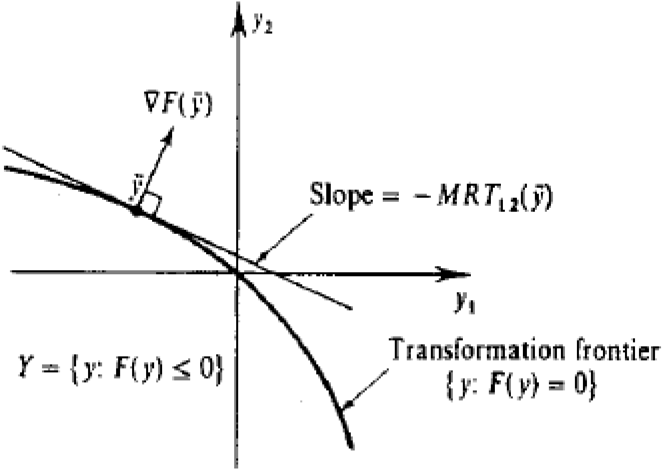
Transformation function

- given the production set Y we can define a *transformation* function.
- it defines the set Y:

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$$

- All points where $F(y) = 0$ are on the boundary of y. $\rightarrow$ **transformation frontier**.
- For example: $-\sqrt{-y_1} + y_2 \leq 0$. y_1 is and input and y_2 is an output.
- But we could have: $-\sqrt{-y_1} + y_2 + y_3 \leq 0$. Two outputs and one input.

Transformation with one input and one output



Marginal rate of transformation

If $F(\cdot)$ is differentiable, and we are at the frontier ($F(\bar{y}) = 0$) then we define *marginal rate of transformation* as:

$$MRT_{lk}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial l}}{\frac{\partial F(\bar{y})}{\partial k}}$$

for each l and k.

What is it? Totally differentiate $F(\bar{y}) = 0$. For $dl \neq 0$ and $dk \neq 0$ and all other d...'s equal to zero:

$$\frac{\partial F(\cdot)}{\partial l} dl + \frac{\partial F(\cdot)}{\partial k} dk = 0$$

Marginal rate of transformation

$$\frac{\partial F(\cdot)}{\partial l} dl + \frac{\partial F(\cdot)}{\partial k} dk = 0$$

rearranging gives:

$$\frac{\frac{\partial F(\cdot)}{\partial l}}{\frac{\partial F(\cdot)}{\partial k}} = -\frac{dk}{dl} = MRT_{lk}(\bar{y})$$

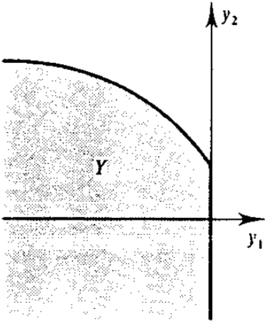
So:

- if both goods are outputs, it shows the tradeoff between production of goods
- if one of goods is an input and other is output, it shows the marginal productivity of an input in terms of an output
- if both goods are inputs, it shows Marginal Rate of Technical Substitution (MRTS) - more on this later.

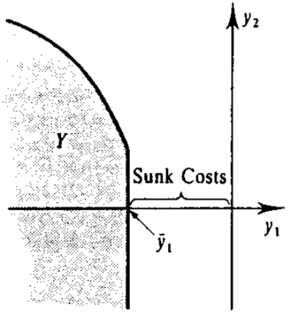
Properties of production sets

- Y is nonempty
- Y is closed (includes its boudary)
- Other properties:
    - No free lunch
    - Possibility of inaction $0 \in Y$ (no sunk costs)
    - Free disposal (extra amounts of inputs can be eliminated at no costs).
    - Irreversibility (you cannot produce inputs from outputs)
    - Non-increasing returns to scale
    - Non-decreasing returns to scale
    - Constant returns to scale.
    - Convexity

Violation of no free lunch (left)

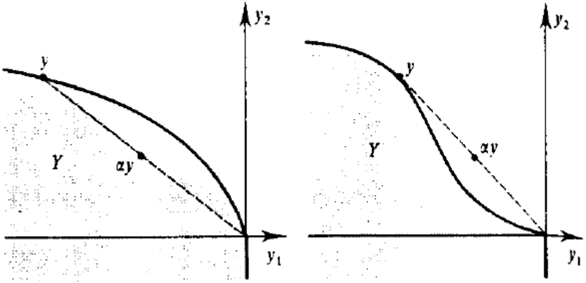


Sunk costs



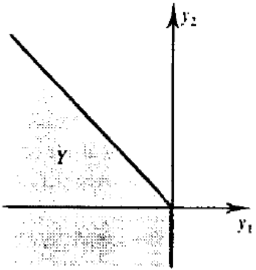
Returns to scale

- Nonincreasing returns to scale: $\alpha y \in Y$, $\alpha < 1$ (convexity - left - any production level can be scaled down)
- Nondecreasing returns to scale: $\alpha y \in Y$, $\alpha > 1$ (any production level can be scaled up)



Constant returns to scale

- Nonincreasing returns to scale: $\alpha y \in Y$, $\alpha > 0$ (convexity - left)



Distinct inputs and outputs

Let $q = (q_1, \ldots, q_M) \geq 0$ be outputs and $z = (z_1, \ldots, z_{L-M}) \geq 0$ be inputs.

In that case we measure inputs and outputs in nonnegative numbers (unlike before)

A commonly use example: one output, many inputs

- We can define a **production function** $f(\cdot)$ that gives rise to the production set

$$Y = \{(-z_1, \ldots, -z_{L-1}, q) : q - f(z_1, \ldots, z_{L-1}) \leq 0\} \text{ and } (z_1, \ldots, z_{L-1}) \geq 0$$

- If we are on boundary: $q = f(z_1, \ldots, z_{L-1})$

MRTS

Given the production function $f(z_1, \ldots, z_{L-1})$, we define the marginal rate of technical substitution for input l and k as:

$$MRTS_{lk}(\bar{z}) = \frac{\frac{\partial f(\bar{z})}{\partial l}}{\frac{\partial f(\bar{z})}{\partial k}}$$

When we start at the level of production $q = f(\bar{z})$ and the initial allocation of inputs $\bar{z}$, then it measures how much we have to add of one input if we take away some of the other

$\rightarrow$ analogy with the consumer theory and MRS.

Returns to scale

- constant: $f(\alpha z) = \alpha f(z)$ for any $\alpha > 0$
- increasing: $f(\alpha z) > \alpha f(z)$ for any $\alpha > 1$
- decreasing: $f(\alpha z) < \alpha f(z)$ for any $\alpha > 1$

Isoquant

Suppose we have just one output q:

Isoquant:

$$Q(y) = \{z \in \mathbb{R}_L^+ : (q, -z) \in Y\}$$

so it shows all combination of inputs that produce a given output q.

Input requirement set - a set of inputs that produce at least the output q. (area above isoquant).

Example: perfect substitutes

with two inputs and one output. You can *subsitute* inputs z_1 and z_2 in combinations given by a_1 and a_2($a's$ are marginal productivities)

$q = f(z) = a_1 z_1 + a_2 z_2$.

Equation of the isoquant at q: $q/a_2 - \frac{a_1}{a_2} z_1 = z_2$.

How does it look like?

What is the MRTS?

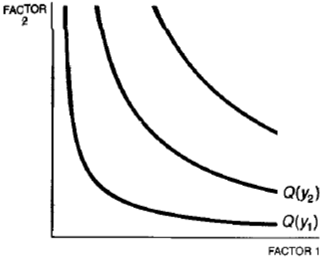
Cobb-Douglas

2 inputs, one output:

$$q = f(z) = z_1^{\alpha} z_2^{1-\alpha}$$

$$Y = \{(q, -z) \in \mathbb{R}^3 : q \leq z_1^{\alpha} z_2^{1-\alpha}\}$$

$$Q(q, -z) = \{(z) \in \mathbb{R}^2_+ : q = z_1^{\alpha} z_2^{1-\alpha}\}$$

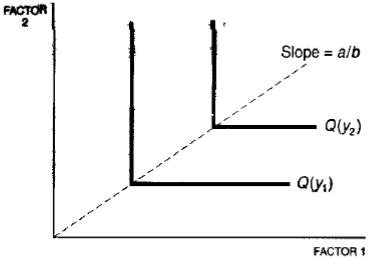


Leontief

2 inputs, one output:

$$q = f(z) = \min(a_1 z_1, a_2 z_2)$$

$\frac{1}{a_i}$ is the *unit factor requirement* - minimal input for one unit of output



$$Y = \{(q, -z) \in \mathbb{R}^3 : q \leq \min(a_1 z_1, a_2$$

$$Q(q, -z) = \{(q, -z) \in \mathbb{R}^2_+ : q = \min(a_1 z_1, a_2 z_2)\}$$

Elasticity of substitution

- percentage change in the MRTS with a change in the ratio of inputs given fixed output
- or: the curvature of isoquant (how the ratio of factor inputs changes as the slope of the isoquant changes).

$$\sigma = \frac{\frac{\Delta(z_2/z_1)}{z_2/z_1}}{\frac{\Delta MRTS}{MRTS}}$$

or in derivative terms: $\sigma = \frac{MRTS}{z_2/z_1}\frac{d(z_2/z_1)}{dMRTS}$.

- or approximation in logs logs:

$$\sigma = \frac{d\ln(z_2/z_1)}{d\ln|MRTS|}$$

- For Cobb-Douglas $\sigma = 1$ - it is (constant elasticity of substitution function)

A more general CES

(2 inputs)

$$q = (a_1 z_1^{\rho} + a_2 z_2^{\rho})^{\frac{1}{\rho}}$$

- constant returns to scale
- $\sigma = \frac{1}{1-\rho}$ is the elasticity of substitution
- in particular, as $\rho \to 0$ the properties of CES tend to Cobb-Douglas