

# Probability Calculus

**Anna Janicka**

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**CHEBYSHEV INEQUALITIES**

**CONVERGENCE**

**LAWS OF LARGE NUMBERS**

# Plan for Today

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1. Conditional expectation as a predictor
2. Chebyshev Inequalities
3. Types of convergence of Random Variables
  - convergence almost surely
  - convergence in probability
4. Laws of Large Numbers
  - Weak LLN
  - Strong LLN



# Conditional Expectation as an approximation

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## 1. Theorem:

*Let  $X, Y : \Omega \rightarrow \mathbb{R}$  be random variables such that  $\mathbb{E}Y^2 < \infty$ . Then, the function  $\varphi^* : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $\varphi^*(x) = \mathbb{E}(Y|X = x)$ , satisfies:*

$$\mathbb{E}(Y - \varphi^*(X))^2$$

$$= \min\{\mathbb{E}(Y - \varphi(X))^2 : \varphi \text{ is a Borel function } : \mathbb{R} \rightarrow \mathbb{R}\}.$$



# Chebyshev Inequality

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1. Sometimes we are only interested in inequalities of the type

$$\mathbb{P}(X \geq x) \leq \alpha.$$

2. Chebyshev inequality

*Let  $X$  be a nonnegative integrable random variable, and let  $\varepsilon > 0$ . We have:*

$$\mathbb{P}(X \geq \varepsilon) \leq \frac{\mathbb{E}X}{\varepsilon}$$



# Chebyshev Inequality – derivatives

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## 3. Chebyshev inequality for $|X|^p$ , $(X - \mathbb{E}X)^2$ and $e^{\lambda X}$

*Let  $X$  be a random variable.*

- **Markov Inequality:** *For any  $p > 0$  such that  $\mathbb{E}|X|^p$  exists, and any  $\varepsilon > 0$ ,*  
$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{\mathbb{E}|X|^p}{\varepsilon^p}.$$
- **Chebyshev-Bienaymé Inequality:** *For any  $\varepsilon > 0$ , if the random variable  $X^2$  is integrable,*  
$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}.$$
- **Exponential Chebyshev Inequality:**  
*Let us assume that  $\mathbb{E}e^{pX} < \infty$  for a given value  $p > 0$ . Then, for any  $\lambda \in [0, p]$  and for any  $\varepsilon > 0$ ,*  
$$\mathbb{P}(X \geq \varepsilon) \leq \frac{\mathbb{E}e^{\lambda X}}{e^{\lambda \varepsilon}}.$$

# Bernstein Inequality

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## 4. Bernstein inequality:

Let  $S_n$  be a random variable from a binomial distribution with parameters  $n$  and  $p$ . Then, for any  $\varepsilon > 0$ , we have

$$\mathbb{P} \left( \left| \frac{S_n}{n} - p \right| \geq \varepsilon \right) \leq 2e^{-2\varepsilon^2 n}.$$

also: 
$$\mathbb{P} \left( \frac{S_n}{n} \geq p + \varepsilon \right) \leq e^{-2\varepsilon^2 n}$$

$$\mathbb{P} \left( \frac{S_n}{n} \leq p - \varepsilon \right) \leq e^{-2\varepsilon^2 n}$$

## 5. Examples

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# Comparison of inequalities

$\varepsilon$	n	Chebyshev- Bienaymé	Bernstein
0,1	100	0,25	0,2707
0,1	1000	0,025	4,1223E-09
0,05	100	1	1,2131
0,05	1000	0,1	0,0135
0,05	10000	0,01	3,8575E-22
0,01	100	25	1,9604
0,01	1000	2,5	1,6375
0,01	10000	0,25	0,2707
0,01	100000	0,025	4,1223E-09



# Types of convergence:

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## 1. Almost sure convergence

A sequence  $(X_n)_{n \geq 1}$  of random variables over  $\Omega$  converges **almost surely** to  $X$ , if  $\mathbb{P}(\lim_{n \rightarrow \infty} X_n = X) = 1$ .

Equivalently, we may say that there exists a subset  $\Omega' \subset \Omega$  such that  $\mathbb{P}(\Omega') = 1$ , such that for any  $\omega \in \Omega'$ , we have  $\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)$ .

Denoted by  $X_n \xrightarrow{a.s.} X$ .

An alternative formulation:

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$$\lim_{n \rightarrow \infty} \mathbb{P}(\sup_{k \geq n} |X_k - X| > \varepsilon) = 0.$$

# Types of convergence – cont.

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## 2. Convergence in probability

A sequence  $(X_n)_{n \geq 1}$  of random variables over  $\Omega$  converges **in probability** to  $X$ , if for any  $\varepsilon > 0$ , we have that  $\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$ .

Equivalently, for any  $\varepsilon > 0$ ,  $\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \leq \varepsilon) = 1$ .

Denoted by  $X_n \xrightarrow{\mathbb{P}} X$  or  $\text{plim}_{n \rightarrow \infty} X_n = X$ .

## 3. Almost sure convergence $\Rightarrow$ convergence in probability

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# Properties of limits of RV

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## 4. Theorem

*Let  $(X_n)_{n \geq 1}$  and  $(Y_n)_{n \geq 1}$  be sequences of random variables. If  $(X_n)_{n \geq 1}$  converges to  $X$  and  $(Y_n)_{n \geq 1}$  converges to  $Y$  almost surely (/in probability), then  $X_n \pm Y_n \rightarrow X \pm Y$  and  $X_n \cdot Y_n \rightarrow XY$  almost surely (/in probability).*



# Weak Laws of Large Numbers

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## 1. Weak Law of Large Numbers for the Bernoulli Scheme

*Let  $X_1, X_2, \dots$  be independent with distributions  $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0)$ . We then have that  $(S_n/n)$  converges in probability to  $p$ ; in other words, for any  $\varepsilon > 0$ , we have  $\lim_{n \rightarrow \infty} \mathbb{P} \left( \left| \frac{S_n}{n} - p \right| > \varepsilon \right) = 0$ .*

$$S_n = X_1 + X_2 + \dots + X_n$$



# Weak Laws of Large Numbers – cont.

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## 2. Weak Law of Large Numbers for uncorrelated random variables

*Let  $X_1, X_2, \dots$  be uncorrelated random variables with a common upper bound to their variances. Then, the sequence  $(X_n)_{n \geq 1}$  satisfies the weak law of large numbers:  $\frac{S_n - \mathbb{E}S_n}{n} \xrightarrow{\mathbb{P}} 0$ , i.e. for any  $\varepsilon > 0$  we have  $\lim_{n \rightarrow \infty} \mathbb{P} \left( \left| \frac{S_n - \mathbb{E}S_n}{n} \right| > \varepsilon \right) = 0$ .*



# Weak Laws of Large Numbers – cont. (2)

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## Examples

- independent events
- variances without bounds  $\rightarrow$  NO
- correlated RV  $\rightarrow$  NO
- embarrassing question



# Strong Laws of Large Numbers

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## 1. Strong Law of Large Numbers for the Bernoulli Scheme

*Let  $X_1, X_2, \dots$  be a sequence of independent random variables, such that*

$$\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0), \quad n = 1, 2, \dots$$

*Then, the sequence  $(S_n/n)$  converges almost surely to  $p$ ; i.e., there exists an event  $\Omega'$  of measure 1 such that for any  $\omega \in \Omega'$ , we have*

$$\lim_{n \rightarrow \infty} \frac{S_n(\omega)}{n} = p.$$



# Strong Laws of Large Numbers – cont.

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## 2. Kolmogorov's Strong Law of Large Numbers

*Let  $X_1, X_2, \dots$  be a sequence of independent, identically distributed integrable random variables. Then,*

$$\frac{S_n}{n} \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E}X_1.$$

flaw: we do not know the rate of convergence

uses: many, e.g. verification of the probabilistic model, MC methods

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# Application of the SLLN:

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## 1. Convergence of the sample mean

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E}X_1.$$

## 2. Convergence of the sample variance

$$S^2 = \frac{1}{n} \sum_{k=1}^n (X_k - \bar{X})^2 \xrightarrow[n \rightarrow \infty]{a.s.} \text{Var} X_1.$$



## Applications of the SLLN – cont.

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3. Convergence of sample distributions: for

$$\mu_n(A) = \frac{1_A(X_1) + 1_A(X_2) + \dots + 1_A(X_n)}{n}$$

we have  $\mu_n(A) \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E}1_A(X_1) = \mathbb{P}(X_1 \in A)$

4. Convergence of sample CDFs: for

$$F_n(t) = \frac{1_{\{X_1 \leq t\}} + 1_{\{X_2 \leq t\}} + \dots + 1_{\{X_n \leq t\}}}{n}$$

we have  $F_n(t) \xrightarrow[n \rightarrow \infty]{a.s.} F(t)$



# Applications of SLLN – cont. (2)

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## 5. Glivenko–Cantelli Theorem

*Let  $X_1, X_2, \dots$  be independent random variables from a distribution with a CDF  $F$ . Then,*

$$\sup_{t \in \mathbb{R}} |F_n(t) - F(t)| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$
