

Probability Calculus

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LINEAR REGRESSION
CONDITIONAL EXPECTATION

Plan for Today

1. Linear regression
2. Conditional expectation



Linear regression

1. Best (in terms of average square deviation) **linear** approximation of variable Y with variable X , i.e. $aX+b$:
minimizes

$$f(a, b) = \mathbb{E}(Y - aX - b)^2$$

solution:

$$a = \frac{\text{Cov}(X, Y)}{\text{Var}X} \quad b = \mathbb{E}Y - \frac{\text{Cov}(X, Y)}{\text{Var}X} \mathbb{E}X$$



Conditional Expectations

1. Intuition

2. Definition in the discrete case:

Let (X, Y) be a discrete random vector such that $\mathbb{E}Y$ exists.

For any $x \in \mathbb{R}$ such that $\mathbb{P}(X = x) > 0$, we define the

conditional expected value of variable Y given

$X = x$ as the expected value of a random variable

with distribution $\mu(A) = \mathbb{P}(Y \in A | X = x)$.

That is, if $S_x = \{y \in \mathbb{R} : \mathbb{P}(X = x, Y = y) > 0\}$, we have

$$\mathbb{E}(Y | X = x) = \sum_{y \in S_x} y \mathbb{P}(Y = y | X = x).$$



Conditional Expected Value of discrete RV

3. Example:

- double 0-1
- function of X

4. Transformations

Let (X, Y) be a discrete random vector, and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ a Borel function such that $\mathbb{E}|\varphi(Y)| < \infty$. We then have that for any x such that $\mathbb{P}(X = x) > 0$:

$$\mathbb{E}(\varphi(Y)|X = x) = \sum_{y \in S_x} \varphi(y) \mathbb{P}(Y = y|X = x),$$

where $S_x = \{y \in \mathbb{R} : \mathbb{P}(X = x, Y = y) > 0\}$.



Conditional density

5. Definition

Let (X, Y) be a continuous random vector with density $g : \mathbb{R}^2 \rightarrow [0, \infty)$. Let $g_X(x) = \int_{-\infty}^{\infty} g(x, y) dy > 0$ be the marginal density of X . For all $x \in \mathbb{R}$, we define **conditional density** of variable Y given

$X = x$ as the function

$$g_{Y|X}(y|x) = \begin{cases} \frac{g(x, y)}{g_X(x)} & \text{if } g_X(x) > 0 \\ f(y) & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R} \rightarrow [0, \infty)$ is any density function of our choice.



Conditional density – cont.

6. Properties:

- density
- corresponds to conditional probability
- different functions possible
- OK for independent variables

7. Examples:

- uniform distribution over square
- “chain rule”



Conditional Expected value of continuous RV

8. Definition

Let (X, Y) be a continuous random vector with density $g : \mathbb{R}^2 \rightarrow [0, \infty)$, such that $\mathbb{E}|Y| < \infty$. For all $x \in \mathbb{R}$ we define the **conditional expected value of variable Y given $X = x$** as the expected value of a random variable with density $f_x(y) = g_{Y|X}(y|x)$, i.e.

$$\mathbb{E}(Y|X = x) = \int_{-\infty}^{\infty} yg_{Y|X}(y|x)dy.$$

9. Example



Conditional Expected value of continuous RV – cont.

10. Transformations:

Let (X, Y) be a continuous random vector with density $g : \mathbb{R}^2 \rightarrow [0, \infty)$, and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function such that $\mathbb{E}|\varphi(Y)| < \infty$. Then, we have that for any $x \in \mathbb{R}$, $\mathbb{E}(\varphi(Y)|X = x) = \int_{-\infty}^{\infty} \varphi(y)g_{Y|X}(y|x)dy$.



Conditional Expectation

11. General definition of conditional expectation

*Let (X, Y) be a random vector, such that $\mathbb{E}|Y| < \infty$. The **conditional expected value of Y given X** , denoted as $\mathbb{E}(Y|X)$, is a random variable such that $\mathbb{E}(Y|X) = m(X)$, where $m(x) = \mathbb{E}(Y|X = x)$.*

12. Examples



Properties of Conditional Expectations

13. Properties of expected values

Let $X, Y, Z : \Omega \rightarrow \mathbb{R}$ be random variables such that $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$. We have:

- (i) *If $X \geq 0$, then $\mathbb{E}(X|Z) \geq 0$.*
- (ii) *$|\mathbb{E}(X|Z)| \leq \mathbb{E}(|X||Z)$.*
- (iii) *For any $a, b \in \mathbb{R}$ we have*
$$\mathbb{E}(aX + bY|Z) = a\mathbb{E}(X|Z) + b\mathbb{E}(Y|Z).$$



Properties of Conditional Expectations – cont.

13. Specific properties

Let $X, Y : \Omega \rightarrow \mathbb{R}$ be random variables such that $\mathbb{E}|Y| < \infty$. We have that

- (i) $\mathbb{E}|\mathbb{E}(Y|X)| < \infty$ and $\mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}Y$.*
- (ii) If X and Y are independent, then $\mathbb{E}(Y|X) = \mathbb{E}Y$.*
- (iii) If $h(X)$ is a limited random variable, then $\mathbb{E}(h(X) \cdot Y|X) = h(X)\mathbb{E}(Y|X)$.*



Conditional Probability

14. Definition

Let X be a random variable. For any event $A \in \mathcal{F}$, we define

$$\mathbb{P}(A|X) = \mathbb{E}(1_A|X)$$



Conditional Expectation as an approximation

1. Theorem:

Let $X, Y : \Omega \rightarrow \mathbb{R}$ be random variables such that $\mathbb{E}Y^2 < \infty$. Then, the function $\varphi^ : \mathbb{R} \rightarrow \mathbb{R}$, such that $\varphi^*(x) = \mathbb{E}(Y|X = x)$, satisfies:*

$$\mathbb{E}(Y - \varphi^*(X))^2$$

$$= \min\{\mathbb{E}(Y - \varphi(X))^2 : \varphi \text{ is a Borel function} : \mathbb{R} \rightarrow \mathbb{R}\}.$$

