Probability Calculus

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RANDOM VECTORS – cont.

INDEPENDENCE OF RANDOM VARIABLES

Plan for Today

1. Marginal distributions – cont.
2. Expected values of functions of random vectors
3. Covariance, correlation
4. Expected value, variance of random vectors
5. Independence of random variables
   - properties and characteristics of independent RV

Marginal distributions – (cont.)

Marginal distributions of continuous RV:

Let (X, Y) be a random vector with density g. The marginal distributions of X and Y are also continuous, and the respective densities are equal to
$$g_X(x) = \int_{\mathbb{R}} g(x,y)dy, \qquad g_Y(y) = \int_{\mathbb{R}} g(x,y)dx.$$

More generally, if an n-dimensional random vector has a joint density function g, then the i-th component is continuous with density g_i, such that
$$g_i(x_i) = \int\int\int_{\mathbb{R}^{n-1}} g(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$
(the integral is over all variables other than X_i).

If marginals are continuous, then the joint distribution need not be

Characteristics of random vectors

1. Expected values of functions of the components of a RV:

(i) Let (X,Y) be a discrete random vector with support S, and let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a Borel function. Then,
$\mathbb{E}\phi(X,Y) = \sum_{(x,y)\in S} \phi(x,y)\mathbb{P}((X,Y) = (x,y))$
(if the sum converges absolutely).
(ii) Let (X,Y) be a continuous random vector with density g and let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a Borel function. Then,
$\mathbb{E}\phi(X,Y) = \iint_{\mathbb{R}^2} \phi(x,y)g(x,y)dxdy$
(if the expected value exists).

2. Examples

The covariance and correlation coefficient

3. Definitions

Let (X,Y) *be a random vector, such that* X *and* Y *have expected values, and such that* $\mathbb{E}|XY| < \infty$. *The* **covariance** *of variables* X *and* Y *is the value*

$\mathrm{Cov}(X,Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y)$.

If, additionally, the variances of the two random variables exist, and $\mathrm{Var}X > 0$ *and* $\mathrm{Var}Y > 0$, *we may define the (Pearson's)* **correlation coefficient** *of variables* X *and* Y *as* $\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X \cdot \mathrm{Var}Y}} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$.

Covariance and correlation coefficient – cont.

4. Properties:
   - invariance to shifts
   - bilinearity of the covariance
   - variance as a special case
   - simplifying formula:
     $$\mathrm{Cov}(X,Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}X \cdot \mathbb{E}Y.$$
   - capture the *linear* relationship, in other cases may be misleading

Correlation coefficient – properties

5. Schwarz inequality

Let $X, Y : \Omega \to \mathbb{R}$ *be random variables such that* $\mathbb{E}X^2 < \infty$ *and* $\mathbb{E}Y^2 < \infty$. *We then have* $|\mathbb{E}XY| \leqslant (\mathbb{E}X^2)^{1/2}(\mathbb{E}Y^2)^{1/2}$.
Furthermore, we have an equality if and only if there exist two numbers $a, b \in \mathbb{R}$ *not simultaneously equal to zero, such that* $\mathbb{P}(aX = bY) = 1$.

6. Consequences for the correlation coef.

Let $X, Y : \Omega \to \mathbb{R}$ *be random variables with finite nonzero variances. Then* $|\rho(X, Y)| \leqslant 1$. *Furthermore, if* $|\rho(X, Y)| = 1$, *then there exist two numbers* $a, b \in \mathbb{R}$, *such that* $Y = aX + b$.

Expected value and covariance matrix

7. Definitions:

Let (X, Y) be a two-dimensional random vector.
Then, we have:
(i) If X and Y have expected values, then the **expected value** $\mathbb{E}(X, Y)$ *of the vector* (X, Y) *is the vector* $(\mathbb{E}X, \mathbb{E}Y)$.
(ii) If X and Y have variances, then the **covariance matrix** *of the vector* (X, Y) *is the matrix*

$$\begin{bmatrix} \mathrm{Var} X & \mathrm{Cov}(X, Y) \\ \mathrm{Cov}(X, Y) & \mathrm{Var} Y \end{bmatrix}$$

For higher dimensions ($\mathbb{R}^d$, $d \geqslant 3$), *we have, similarly:*
the expected value is the vector $(\mathbb{E}X_1, \mathbb{E}X_2, \ldots, \mathbb{E}X_d)$,
and the covariance matrix is the matrix $(\mathrm{Cov}(X_i, X_j))_{1 \leqslant i,j \leqslant d}$.

Properties of EX and the covariance matrix

8. Linearity

Let $X = (X_1, X_2, \ldots, X_n)$ *be a random vector of dimension* n, *and* A – *a* $m \times n$ *matrix.*
(i) If X *has a finite expected value, then* AX *also has a finite expected value, and* $\mathbb{E}(AX) = A\mathbb{E}X$.
(ii) If the covariance matrix Q_X *of the vector* X *exists, then there exists also the covariance matrix of the vector* AX, *and it is equal to* $Q_{AX} = AQ_XA^t$.

Independent RV

1. Definition of independence

Variables $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ *are* **independent**, *if for any sequence of Borel sets* $B_1, B_2, \ldots, B_n$, *we have*

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n)$$
$$= \mathbb{P}(X_1 \in B_1) \cdot \mathbb{P}(X_2 \in B_2) \cdot \ldots \cdot \mathbb{P}(X_n \in B_n).$$

2. Independence of discrete RV

Let $X_1, X_2, \ldots, X_n$ *be discrete random variables with supports* S_{X_i}, *respectively. In this case,* $X_1, X_2, \ldots, X_n$ *are independent if and only if for any sequence* $x_1,\ x_2,\ \ldots,\ x_n$ *such that* $x_i \in S_{X_i}$, $i = 1, 2, \ldots, n$, *we have*

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$
$$= \mathbb{P}(X_1 = x_1) \cdot \mathbb{P}(X_2 = x_2) \cdot \ldots \cdot \mathbb{P}(X_n = x_n).$$

Independent RV – cont.

3. Example

4. Independence of continuous RV

Let $X_1, X_2, \ldots, X_n : \Omega \to \mathbb{R}$ *be continuous random variables with probability densities* $g_1, g_2, \ldots, g_n$, *respectively. In this case,* $X_1, X_2, \ldots, X_n$ *are independent if and only if* $g\colon \mathbb{R}^n \to [0, \infty)$, *defined as*

$$g(x_1, x_2, \ldots, x_n) = g_1(x_1) \cdot g_2(x_2) \cdot \ldots \cdot g_n(x_n),$$

is a density function of the distribution $\mu_{(X_1, X_2, \ldots, X_n)}$.

5. Examples

- uniform distribution on square
- uniform distribution on circle

Independent RV – cont. (2)

6. Transformations of RV

Let $X_{1,1}, X_{1,2}, \ldots, X_{1,k_1}, X_{2,1}, X_{2,2}, \ldots, X_{2,k_2}, \ldots, X_{n,1}, X_{n,2}, \ldots, X_{n,k_n}$ *be independent random variables, and* $\varphi_i : \mathbb{R}^{k_i} \to \mathbb{R}$, $i = 1, 2, \ldots, n$ *be Borel functions. We then have that the variables*

$$
\begin{aligned}
Y_1 &= \varphi_1(X_{1,1}, X_{1,2}, \ldots, X_{1,k_1}), \\
Y_2 &= \varphi_2(X_{2,1}, X_{2,2}, \ldots, X_{2,k_2}), \\
&\ldots \\
Y_n &= \varphi_n(X_{n,1}, X_{n,2}, \ldots, X_{n,k_n})
\end{aligned}
$$

are independent.

Properties of independent RV

2. Expected value of product

Let $X_1, X_2, \ldots, X_n$ *be independent random variables with expected values. Then, the variable* $X = X_1 \cdot X_2 \cdot \ldots \cdot X_n$ *also has an expected value, and we have*

$$\mathbb{E}X = \mathbb{E}(X_1 \cdot X_2 \cdot \ldots \cdot X_n) = \mathbb{E}X_1 \cdot \mathbb{E}X_2 \cdot \ldots \cdot \mathbb{E}X_n.$$

3. Example

4. Covariance of independent RV

Let X *and* Y *be independent random variables, such that* $\mathbb{E}|XY| < \infty$. *We then have* $\mathrm{Cov}(X, Y) = 0$.

5. Non-correlation

Properties of independent RV – cont.

6. One-way implication only!

independence $\Rightarrow$ non-correlation but

$\Leftarrow$ IS NOT TRUE!

7. Example – uniform distribution on circle

8. Sum of variances

Let $X_1, X_2, \ldots, X_n$ *be* ~~*independent*~~ *random variables with finite variances. Then, the variable* $X = X_1 + X_2 + \ldots + X_n$ *also has a finite variance, and we have*

$$\operatorname{Var} X = \operatorname{Var}(X_1 + X_2 + \ldots + X_n)$$
$$= \operatorname{Var}(X_1) + \operatorname{Var}(X_2) + \ldots + \operatorname{Var}(X_n).$$
$$+ 2\sum_{i<j} \operatorname{Cov}(X_i, X_j).$$

Properties of independent RV – cont. (2)

9. Example – sum of points on dice