

Probability Calculus

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RANDOM VARIABLES – CONT.

**RANDOM VECTORS – MULTIDIMENSIONAL RANDOM
VARIABLES**

Plan for Today

1. Moments of random variables
2. Sample characteristics
3. Definition of a Random Vector
4. Joint and marginal distributions
5. Discrete and continuous RV
6. Expected values of functions
7. Covariance, correlation
8. Expected value, variance



Moments – reminder

1. Definitions

For $p \in (0, \infty)$, we define:

(i) the **absolute moment** of rank p for random variable X as $\mathbb{E}|X|^p$ (if this value is finite);

For $p \in \mathbb{N}$, we define:

(ii) the **moment** of rank p for random variable X as $\mathbb{E}X^p$ (provided that the p -th absolute moment exists);

(iii) the **central moment** of rank p for random variable X as $\mathbb{E}(X - \mathbb{E}X)^p$ (provided that the p -th absolute moment exists).



Moments: skewness, kurtosis

2. Definitions

Let X be a random variable such that $\mathbb{E}|X|^3 < \infty$.

The **skewness** of X is

$$\alpha_3 = \frac{\mathbb{E}(X - \mathbb{E}X)^3}{(D^2X)^{3/2}} = \frac{\mathbb{E}(X - \mathbb{E}X)^3}{\sigma_X^3}.$$

Let X be a random variable such that $\mathbb{E}|X|^4 < \infty$.

The **kurtosis** of X is

$$\alpha_4 = \frac{\mathbb{E}(X - \mathbb{E}X)^4}{(D^2X)^2} - 3 = \frac{\mathbb{E}(X - \mathbb{E}X)^4}{\sigma_X^4} - 3.$$

3. Example: standard normal distribution



Empirical distributions

1. In reality, we frequently do not know the distributions of random variables, and work with *samples* instead.

2.

*Let X_1, X_2, \dots, X_n be random variables with unknown distributions. An **Empirical distribution (measure)** for this sample is*

$$\mu_n(A) = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}(A) = \frac{|\{i \leq n : X_i \in A\}|}{n},$$



Empirical distributions – cont.

3. An empirical distribution function of the sample X_1, X_2, \dots, X_n is the function $F: \mathbb{R} \rightarrow [0, 1]$, such that
- $$F_n(t) = \mu_n((-\infty, t]) = \frac{|\{i \leq n: X_i \leq t\}|}{n}.$$

this is the CDF of the empirical distribution

4. A Quantile of rank p of the sample X_1, \dots, X_n is any number x_p , such that
- $$\mu_n((-\infty, x_p]) \geq p$$
- $$\mu_n([x_p, \infty)) \geq 1 - p.$$



Empirical distributions – cont (2)

5. A Sample mean for X_1, X_2, \dots, X_n is equal to $m = \frac{X_1 + X_2 + \dots + X_n}{n}$,
i.e. the arithmetic mean of X_1, X_2, \dots, X_n .

6. A sample variance for X_1, X_2, \dots, X_n is equal to $s^2 = \frac{1}{n} \sum_{i=1}^n (X_i - m)^2$,
where m is the sample mean.



the mean and the variance of the empirical distribution



End of material for the midterm test
to be held on
Tuesday, December 3rd, 6:45 PM
Aud Max



Random vectors

1. A random vector (X_1, X_2, \dots, X_n)
2. The joint distribution of a random vector:

The (joint) distribution of a random vector $X = (X_1, X_2, \dots, X_n)$ is a probability measure μ_X defined over $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, such that $\mu_X(A) = \mathbb{P}(X \in A)$.

3. Marginal distributions:

$$\mu_{X_i}(B) = \mathbb{P}(X_i \in B) \text{ for } B \subseteq \mathbb{R},$$

such that for $A = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{i-1} \times B \times \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n-i}$

we have



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$$\mathbb{P}(X_i \in B) = \mathbb{P}((X_1, X_2, \dots, X_n) \in A) = \mu_X(A).$$

Random vectors – cont.

4. Example: joint distribution is more than the aggregate of marginal distributions.
5. Cumulative distribution function:

The cumulative distribution function of a random vector (X, Y) is a function $F_{(X,Y)} : \mathbb{R}^2 \rightarrow [0, 1]$, such that $F_{(X,Y)}(s, t) = \mathbb{P}(X \leq s, Y \leq t)$.

6. No simple definitions of quantiles...



Random vectors – types.

7. A discrete RV

A random vector (X, Y) is **discrete**, if there exists a countable set $S \subseteq \mathbb{R}^2$, such that $\mu_{(X,Y)}(S) = 1$.

8. Components are also discrete, marginals obtained by summation

9. A continuous RV

A random vector (X, Y) is **continuous**, if there exists a density function, i.e. a function $g : \mathbb{R}^2 \rightarrow [0, \infty)$, such that for any $A \in \mathcal{B}(\mathbb{R}^2)$, we have $\mu_{(X,Y)}(A) = \iint_A g(x, y) dx dy$.



Random vectors – types (cont.)

10. Examples of continuous RV:

- drawing from a unit square
- drawing from a circle
- a different type of density

11. Marginal distributions of continuous RV:

Let (X, Y) be a random vector with density g . The marginal distributions of X and Y are also continuous, and the respective densities are equal to

$$g_X(x) = \int_{\mathbb{R}} g(x, y) dy, \quad g_Y(y) = \int_{\mathbb{R}} g(x, y) dx.$$



Random vectors – types cont (2).

11. Marginal distributions (cont.)

More generally, if an n -dimensional random vector has a joint density function g , then the i -th component is continuous with density g_i , such that

$$g_i(x_i) = \iiint_{\mathbb{R}^{n-1}} g(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_{i-1} dx_{i+1} \dots dx_n$$

(the integral is over all variables other than X_i).

12. If marginals are continuous, then the joint distribution need not be.



Characteristics of random vectors

13. Expected values of functions of the components of a RV:

(i) Let (X, Y) be a discrete random vector with support S , and let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Borel function. Then,
$$\mathbb{E}\phi(X, Y) = \sum_{(x,y) \in S} \phi(x, y) \mathbb{P}((X, Y) = (x, y))$$
(if the sum converges absolutely).

(ii) Let (X, Y) be a continuous random vector with density g and let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Borel function. Then,
$$\mathbb{E}\phi(X, Y) = \iint_{\mathbb{R}^2} \phi(x, y) g(x, y) dx dy$$
(if the expected value exists).

14. Examples



The covariance and correlation coefficient

15. Definitions

Let (X, Y) be a random vector, such that X and Y have expected values, and such that $\mathbb{E}|XY| < \infty$. The **covariance** of variables X and Y is the value

$$\text{Cov}(X, Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y).$$

If, additionally, the variances of the two random variables exist, and $\text{Var}X > 0$ and $\text{Var}Y > 0$, we may define the (Pearson's) **correlation coefficient** of variables X and Y

$$\text{as } \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}X \cdot \text{Var}Y}} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$



Covariance and correlation coefficient – cont.

16. Properties:

- invariance to shifts
- bilinearity of the covariance
- variance as a special case
- simplifying formula:

$$\text{Cov}(X, Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}X \cdot \mathbb{E}Y.$$

- capture the *linear* relationship, in other cases may be misleading



Correlation coefficient – properties

17. Schwarz inequality

Let $X, Y : \Omega \rightarrow \mathbb{R}$ be random variables such that $\mathbb{E}X^2 < \infty$ and $\mathbb{E}Y^2 < \infty$. We then have

$$|\mathbb{E}XY| \leq (\mathbb{E}X^2)^{1/2}(\mathbb{E}Y^2)^{1/2}.$$

Furthermore, we have an equality if and only if there exist two numbers $a, b \in \mathbb{R}$ not simultaneously equal to zero, such that $\mathbb{P}(aX = bY) = 1$.

18. Consequences for the correlation coef.

Let $X, Y : \Omega \rightarrow \mathbb{R}$ be random variables with finite nonzero variances. Then $|\rho(X, Y)| \leq 1$. Furthermore, if $|\rho(X, Y)| = 1$, then there exist two numbers $a, b \in \mathbb{R}$, such that $Y = aX + b$.



Expected value and covariance matrix

19. Definitions:

Let (X, Y) be a two-dimensional random vector.

Then, we have:

*(i) If X and Y have expected values, then the **expected value** $\mathbb{E}(X, Y)$ of the vector (X, Y) is the vector $(\mathbb{E}X, \mathbb{E}Y)$.*

*(ii) If X and Y have variances, then the **covariance matrix** of the vector (X, Y) is the matrix*

$$\begin{bmatrix} \text{Var}X & \text{Cov}(X, Y) \\ \text{Cov}(X, Y) & \text{Var}Y \end{bmatrix}$$

For higher dimensions (\mathbb{R}^d , $d \geq 3$), we have, similarly: the expected value is the vector $(\mathbb{E}X_1, \mathbb{E}X_2, \dots, \mathbb{E}X_d)$, and the covariance matrix is the matrix $(\text{Cov}(X_i, X_j))_{1 \leq i, j \leq d}$.



