Probability Calculus

Anna Janicka

lecture VII, 19.11.2019

EXPECTED VALUE – CONT.

VARIANCE

MOMENTS

EMPIRICAL DISTRIBUTIONS

Plan for Today

1. Calculating expected values – cont.
2. Variance
3. Moments
4. Empirical distributions

Expected value of a function of a RV

1. Theorem

Let $\phi : \mathbb{R} \to \mathbb{R}$ *be a Borel function.*
(i) If X *is discrete, concentrated on the set* S*, and* $p_x = \mathbb{P}(X = x)$ *for* $x \in S$*, then the random variable* $\phi(X)$ *has an expected value if and only if* $\sum_{x \in S} |\phi(x)| p_x < \infty$*, and the expected value is equal to* $\mathbb{E}\phi(X) = \sum_{x \in S} \phi(x) p_x$.
(ii) If X *is continuous with density* g*, then the random variable* $\phi(X)$ *has an expected value if and only if* $\int_R |\phi(x)| g(x) dx < \infty$*, and the expected value is equal to* $\mathbb{E}\phi(X) = \int_R \phi(x) g(x) dx$.

2. Examples

Expected value of a non-negative RV

3. Calculating EX based on the CDF:

for non-negative integer values

$$\mathbb{E}X = \sum_{k=0}^{\infty} k\mathbb{P}(X=k) = \sum_{k=1}^{\infty} k\mathbb{P}(X=k).$$

$$\begin{aligned} = \;& \mathbb{P}(X=1) + \\ & \mathbb{P}(X=2) + \; \mathbb{P}(X=2) + \\ & \mathbb{P}(X=3) + \; \mathbb{P}(X=3) + \; \mathbb{P}(X=3) + \\ & \mathbb{P}(X=4) + \; \mathbb{P}(X=4) + \; \mathbb{P}(X=4) + \; \mathbb{P}(X=4) + \\ & \ldots \end{aligned}$$

and eventually:

$$\mathbb{E}X = \sum_{k=1}^{\infty} \mathbb{P}(X \geqslant k) = \sum_{k=0}^{\infty} \mathbb{P}(X > k).$$

Expected value – cont (2).

4. Calculating EX based on the CDF – general case of non-negative RV

Let X be a non-negative random variable.
(i) If $\int_0^\infty \mathbb{P}(X > t)dt < \infty$, then X has an expected value and $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt$.
(ii) If $p \in (0, \infty)$ and $\int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt < \infty$, then X^p has an expected value and
$\mathbb{E}X^p = \int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt.$

5. Examples
- geometric distribution
- exponential distribution
- *p*-th moments
- non-discrete non-continuous RV

Variance

1. Definition

Let X be a random variable such that $\mathbb{E}|X| < \infty$ and $\mathbb{E}(X - \mathbb{E}X)^2 < \infty$. The **variance** *of X is defined as $D^2X = VarX = \mathbb{E}(X - \mathbb{E}X)^2$.*
The **standard deviation** *of variable X is the square root of the variance: $\sigma_X = \sqrt{D^2X}$.*

2. Properties

- depends on distribution only
- exists if single condition on EX^2, if limited
- simplified calculations: $D^2X = \mathbb{E}X^2 - (\mathbb{E}X)^2$
- interpretation

Variance – cont.

3. Examples:
   - interpretation
   - die roll
   - uniform distribution
4. Properties, theorem:

Let X be a random variable with a variance.
(i) $D^2X \geqslant 0$, *and the equality holds if and only if there exists a value* $a \in \mathbb{R}$ *such that* $\mathbb{P}(X = a) = 1$.
(ii) $D^2(bX) = b^2 D^2X$ *for any* $b \in \mathbb{R}$.
(iii) $D^2(X + c) = D^2X$ *for any* $c \in \mathbb{R}$.

Variance – cont. (2)

5. Parameters of the normal distribution:
   $N(m, \sigma^2)$

variance

mean

Moments

1. Definitions

For $p \in (0, \infty)$, *we define:*
(i) the **absolute moment** *of rank* p *for random variable* X *as* $\mathbb{E}|X|^p$ *(if this value is finite);*
For $p \in \mathbb{N}$, *we define:*
(ii) the **moment** *of rank* p *for random variable* X *as* $\mathbb{E}X^p$ *(provided that the p-th absolute moment exists);*
(iii) the **central moment** *of rank* p *for random variable* X *as* $\mathbb{E}(X - \mathbb{E}X)^p$ *(provided that the p-th absolute moment exists).*

Moments: skewness, kurtosis

2. Definitions

Let X be a random variable such that $\mathbb{E}|X|^3 < \infty$.
The **skewness** *of X is*

$$\alpha_3 = \frac{\mathbb{E}(X-\mathbb{E}X)^3}{(D^2X)^{3/2}} = \frac{\mathbb{E}(X-\mathbb{E}X)^3}{\sigma_X^3}.$$

Let X be a random variable such that $\mathbb{E}|X|^4 < \infty$.
The **kurtosis** *of X is*

$$\alpha_4 = \frac{\mathbb{E}(X-\mathbb{E}X)^4}{(D^2X)^2} - 3 = \frac{\mathbb{E}(X-\mathbb{E}X)^4}{\sigma_X^4} - 3.$$

3. Example: standard normal distribution

Empirical distributions

1. In reality, we frequently do not know the distributions of random variables, and work with *samples* instead.

2. *Let* $X_1, X_2, \ldots, X_n$ *be random variables with unknown distributions. An* **Empirical distribution (measure)** *for this sample is* $\mu_n(A) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}(A) = \frac{|\{i \leqslant n:\, X_i \in A\}|}{n}$,

Empirical distributions – cont.

3. *An* **empirical distribution function** *of the sample* $X_1, X_2, \ldots, X_n$ *is the function* $F\colon \mathbb{R} \to [0,1]$, *such that*
$$F_n(t) = \mu_n((-\infty, t]) = \frac{|\{i \leqslant n\colon X_i \leqslant t\}|}{n}.$$

this is the CDF of the empirical distribution

4. *A* **Quantile** *of rank* p *of the sample* $X_1, \ldots, X_n$ *is any number* x_p, *such that*
$$\mu_n((-\infty, x_p]) \geqslant p$$
$$\mu_n([x_p, \infty)) \geqslant 1 - p.$$

Empirical distributions – cont (2)

5. *A* **Sample mean** *for* $X_1, X_2, \ldots, X_n$ *is equal to* $m = \frac{X_1+X_2+\ldots+X_n}{n}$, *i.e. the arithmetic mean of* $X_1, X_2, \ldots, X_n$.

6. *A* **sample variance** *for* $X_1, X_2, \ldots, X_n$ *is equal to* $s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - m)^2$, *where* m *is the sample mean.*

the mean and the variance of the empirical distribution