Probability Calculus 2019/2020
Problem set 7

1. Let X be a random variable with density $g(x) = \frac{4}{\pi}\frac{1}{1+x^2}1_{[0,1]}(x)$. Calculate $\mathbb{E}X$ and $\mathbb{E}(3-2X)$.

2. Let X be a random variable with density $g(x) = \frac{1}{2}\sin x 1_{[0,\pi]}(x)$. Calculate $\mathbb{E}X$ and $\mathbb{E}\cos X$.

3. Let X be a standard normal variable. Calculate $\mathbb{E}e^{2X}$ and $\mathbb{E}e^{X^2/4}$.

4. Let X be a random variable describing the weekly income of a worker from a given factory, with a cumulative distribution function of

$$F(t) = \begin{cases} 0 & \text{for } t < 200, \\ ct^2(1500-t) & \text{for } 200 \leqslant t < 1000, \\ 1 & \text{for } t \geqslant 1000, \end{cases}$$

   where $c = 2 \cdot 10^{-9}$. Calculate the mean income of a worker.

5. Let X be a random variable with a Poisson distribution with parameter λ. Calculate $\mathbb{E}X$, $\mathbb{E}X(X-1)$, $\mathbb{E}X^2$ and $\mathbb{E}2^X$.

6. Each edge and each diagonal of a hexagon is either colored red, blue or green. Let X denote the number of triangles with vertices in the hexagon's vertices that are colored with a single color. Calculate $\mathbb{E}X$.

7. We roll a die until we obtain each possible result. Find the mean number of rolls.

8. There are n students in a group. One day, the lecturer distributed graded tests randomly (he gave one test to each student). Let X denote the number of students who got their own test. Find $\mathbb{E}X$.

Some additional problems

Theory (you should know coming into this class):

1. The definition of the expected value of a continuous random variable X.

2. Calculation of the expected value of a transformation of a random variable.

Problems (you should know how to solve after this class)

1. Let X be a random variable from a standard normal distribution. Calculate $\mathbb{E}X(X+1)$ and $\mathbb{E}e^{3X^2/8}$.

2. Let X be a random variable with density $g(x) = (e-1)^{-1}e^{1-x}1_{[0,1]}(x)$. Find $\mathbb{E}(X+1)$ and $\mathbb{E}2^{X+2}$.

3. Let X be a random variable from a geometric distribution with parameter p ($\mathbb{P}(X = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$). Find $\mathbb{E}\min\{X, 100\}$.

4. From the set $\{1, 2, \ldots, 49\}$ we randomly draw 6 numbers without replacement. Let X signify the number of odd numbers. Find $\mathbb{E}X$.

5. We have two light bulbs of type I and three light bulbs of type II. The working life of a light bulb of type I has an exponential distribution with parameter 1, and of light bulb 2 - an exponential distribution with parameter $\frac{1}{2}$. In case a light bulb burns out in our lamp, we change it. Let X denote the total time a lamp works until the supply of light bulbs is finished. Find $\mathbb{E}X$.

6. 10 girls and 10 boys are randomly paired. Calculate the expected value of the number of pairs consisting of girls only.