

Probability Calculus

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CUMULATIVE DISTRIBUTION FUNCTION:QUANTILES.

EXPECTED VALUE

Plan for today

- Quantiles
- Expected value for discrete random variables
- Expected value for continuous random variables
- Properties of the EX operator



Quantiles

1. Definition

Let X be a random variable and $p \in [0, 1]$.

A quantile of rank p of the variable X is any value x_p , such that

$$\mathbb{P}(X \leq x_p) \geq p \text{ and}$$

$$\mathbb{P}(X \geq x_p) \geq 1 - p.$$

2. Examples

- continuous distribution ($N(0,1)$)
- discrete distribution



Expected value – discrete RV

1. Motivation & intuition
2. Definition of expected value for discrete RV

*Let X be a random variable with a discrete distribution, concentrated on $S \subset \mathbb{R}$, and let $p_x = \mathbb{P}(X = x)$ for $x \in S$. We will say that the expected value of X is finite if $\sum_{x \in S} |x|p_x < \infty$. Then we can define this **expected value** of X as $\mathbb{E}X = \sum_{x \in S} xp_x$.*

mean value, depends on the distribution only for a finite set S , the $\mathbb{E}X$ always exists



Expected value – discrete RV. cont.

3. Examples of calculations

- single-valued RV
- die roll
- Binomial distribution (n,p)
- variables without EX:
 - series does not converge at all
 - series does not converge absolutely



Expected value – continuous RV

4. Definition of expected value for continuous RV

Let X be a random variable with density g .

If $\int_{\mathbb{R}} |x|g(x)dx < \infty$, then we will say the expected value of X exists. We define this expected value of X as $\mathbb{E}X = \int_{\mathbb{R}} xg(x)dx$.

5. EX for a limited RV always exists

6. Examples of calculations

- uniform distribution
- standard normal distribution
- ~~■ Cauchy distribution – does not have an EX~~



Expected value – properties (all RV)

7. Properties of $\mathbb{E}X$

Let X and Y be random variables with expected values.

(i) If $X \geq 0$, then $\mathbb{E}X \geq 0$.

(ii) If $X \leq Y$, then $\mathbb{E}X \leq \mathbb{E}Y$.

(iii) $\mathbb{E}X \leq \mathbb{E}|X|$.

(iv) If $a, b \in \mathbb{R}$, then $aX + bY$ has an expected value and $\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$.

(v) If $X = 1_A$, then $\mathbb{E}X = \mathbb{P}(A)$.

8. Generalization of (iv)

9. Examples



Expected value of a function of a RV

10. Theorem

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function.

- (i) If X is discrete, concentrated on the set S , and $p_x = \mathbb{P}(X = x)$ for $x \in S$, then the random variable $\phi(X)$ has an expected value if and only if $\sum_{x \in S} |\phi(x)| p_x < \infty$, and the expected value is equal to $\mathbb{E}\phi(X) = \sum_{x \in S} \phi(x) p_x$.
- (ii) If X is continuous with density g , then the random variable $\phi(X)$ has an expected value if and only if $\int_{\mathbb{R}} |\phi(x)| g(x) dx < \infty$, and the expected value is equal to $\mathbb{E}\phi(X) = \int_{\mathbb{R}} \phi(x) g(x) dx$.

11. Examples



Expected value of a non-negative RV

12. Calculating $\mathbb{E}X$ based on the CDF: for non-negative integer values

$$\begin{aligned}\mathbb{E}X &= \sum_{k=0}^{\infty} k\mathbb{P}(X = k) = \sum_{k=1}^{\infty} k\mathbb{P}(X = k). \\ &= \mathbb{P}(X = 1) + \\ &\quad \mathbb{P}(X = 2) + \mathbb{P}(X = 2) + \\ &\quad \mathbb{P}(X = 3) + \mathbb{P}(X = 3) + \mathbb{P}(X = 3) + \\ &\quad \mathbb{P}(X = 4) + \mathbb{P}(X = 4) + \mathbb{P}(X = 4) + \mathbb{P}(X = 4) + \\ &\quad \dots\end{aligned}$$

and eventually:

$$\mathbb{E}X = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) = \sum_{k=0}^{\infty} \mathbb{P}(X > k).$$



Expected value – cont

13. Calculating EX based on the CDF – general case of non-negative RV

Let X be a non-negative random variable.

(i) If $\int_0^\infty \mathbb{P}(X > t)dt < \infty$, then X has an expected value and $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt$.

(ii) If $p \in (0, \infty)$ and $\int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt < \infty$, then X^p has an expected value and $\mathbb{E}X^p = \int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt$.

14. Examples

- geometric distribution
- exponential distribution
- p -th moments
- non-discrete non-continuous RV



Expected value: summary

1. Mean value
2. For discrete RV: weighted average
3. For continuous RV: average weighted by density
4. Linear operator
5. Calculations for non-negative RV
6. Calculating $E\phi(X)$





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