

Probability Calculus

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INTRODUCTION TO RANDOM VARIABLES

Plan for today

Poisson Theorem – cont.

Introduction to real-valued random variables

- Definition
 - Examples of discrete and continuous random variables
 - Definition of the distribution of a random variable
 - Description of the distribution of a random variable – examples
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- Cumulative Distribution Function



Poisson Theorem – cont.

1. Poisson Theorem

*If $p_n \in [0, 1]$, $\lim_{n \rightarrow \infty} np_n = \lambda > 0$,
then for $k = 0, 1, 2, \dots$,*

we have that

$$\lim_{n \rightarrow \infty} \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$$

2. Assessment of approximation error

*Let S_n denote the number of successes in a
Bernoulli process with n trials and a
probability of success in a single trial equal to p ,
and let $\lambda = np$. For any $A \subset \{0, 1, 2, \dots\}$, we have*

$$\left| \mathbb{P}(S_n \in A) - \sum_{k \in A} \frac{\lambda^k}{k!} e^{-\lambda} \right| \leq \frac{\lambda^2}{n}.$$

Random variables – basics

1. Motivation – functions of the results of an experiment

2. Definition of a **random variable**

A real-valued random variable is any function $X : \Omega \rightarrow \mathbb{R}$, such that for all $a \in \mathbb{R}$ the set $X^{-1}((-\infty, a])$ is an event, i.e. $X^{-1}((-\infty, a]) \in \mathcal{F}$.

$$X^{-1}((-\infty, a]) = \{\omega \in \Omega : X(\omega) \leq a\}$$

3. Examples

- number of heads
- sum of points on dice

■ the distance to a given point



Random variables – distribution

4. Functions of random variables
5. Examples of descriptions of random variables.
6. Definition of a random v. **distribution**

The probability distribution of a random variable X (real-valued) is the probability μ_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that $\mu_X(A) = \mathbb{P}(X \in A)$.

7. Different r.v. have the same distributions
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notation: $X \sim \mu$

we forget about Ω

Random variables – examples

8. Examples of random variables

- die roll
- **discrete distributions**
- Binomial distribution
- Geometric distribution
- Poisson distribution
- uniform distribution over an interval: a **continuous distribution**
- another continuous distribution



Continuous random variables

9. Definition of a **continuous random variable** and a **density function**

*A random variable X has a **continuous distribution**, if there exists a function $g : \mathbb{R} \rightarrow \mathbb{R}_+$, such that for any set $A \in \mathcal{B}(\mathbb{R})$,*
$$\mu_X(A) = \mathbb{P}(X \in A) = \int_A g(x) dx.$$
 *g is called the **probability density function** of X .*

10. The properties of density functions

- nonnegative
- normalized

■ determines the distribution unequivocally



Random variable examples – cont.

11. More examples of continuous random variables

- uniform distribution
- exponential distribution
- standard normal distribution
- general normal distribution
- (Dirac delta)



Random variables – the CDF

12. The definition of a CDF

The Cumulative distribution function

of a random variable $X : \Omega \rightarrow \mathbb{R}$

is a function $F_X : \mathbb{R} \rightarrow [0, 1]$, such that

$$F_X(t) = \mathbb{P}(X \leq t).$$

depends on the distribution only!
→ CDF of distribution

13. Examples of CDFs

- Dirac delta
 - Two-point distribution – discrete distribution
 - Exponential distribution
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