Probability Calculus

Anna Janicka

lecture III, 22.10.2019

INDEPENDENCE OF EVENTS

BERNOULLI PROCESS

POISSON THEOREM

Plan for today

1. Independence of events
2. The Bernoulli Process
3. Approximation of the Bernoulli Process for large n – Poisson Theorem

Independence of Events

1. Definition

$$\textit{Events } A \textit{ and } B \textit{ are } \textbf{independent}, \quad \textit{if } \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot P(B).$$

Symmetric.
Stochastic independence

2. Examples
   - die roll
   - choosing a card

Independence of Events – cont.

3. Independence of 3+ events

Events $A_1, A_2, \ldots, A_n$ *are* **independent**, *if for all indices* $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, $k = 2, 3, \ldots, n$, *we have*

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdot \ldots \cdot \mathbb{P}(A_{i_k}).$$

4. Examples.

- The definition may not be simplified!
- Independence and pairwise independence

Independence of Events – cont. (2)

5. Theorem. Independence conditions

Let $A_1, A_2, \ldots, A_n$ *be a sequence of events, and denote* $A_i^0 = A_i, A_i^1 = A_i'$.
The following conditions are equivalent:

(i) *events* $A_1, A_2, \ldots, A_n$ *are independent,*

(ii) *for any sequence* $\varepsilon_1, \ldots, \varepsilon_n$, *where* $\varepsilon_i \in \{0, 1\}$ $(i = 1, \ldots, n)$, *events* $B_1 = A_1^{\varepsilon_1}, \ldots, B_n = A_n^{\varepsilon_n}$ *are independent,*

(iii) *for any sequence* $\varepsilon_1, \ldots, \varepsilon_n$, *where* $\varepsilon_i \in \{0, 1\}$ $(i = 1, \ldots, n)$, *we have* $\mathbb{P}(A_1^{\varepsilon_1} \cap \ldots \cap A_n^{\varepsilon_n}) = \mathbb{P}(A_1^{\varepsilon_1}) \cdot \ldots \cdot \mathbb{P}(A_n^{\varepsilon_n})$.

Bernoulli Process

1. Definition

A **Bernoulli process** *is a sequence of* n *independent repetitions of a single experiment (referred to as a* Bernoulli trial*) with two possible outcomes: one of these outcomes is referred to as a* success *(usually denoted as 1), and occurs with probability* $p \in [0, 1]$, *and the other one is a failure (usually denoted as 0), and occurs with probability* $q = 1 - p$.

- a finite or an infinite process

Bernoulli Process – cont.

2. Examples
3. Probability in a Bernoulli process:

$$\Omega = \{(a_1, a_2, \ldots, a_n) : a_i \in \{0,1\}, i = 1,\, 2\, \ldots,\, n\}$$

$$\mathcal{F} = 2^{\Omega}$$

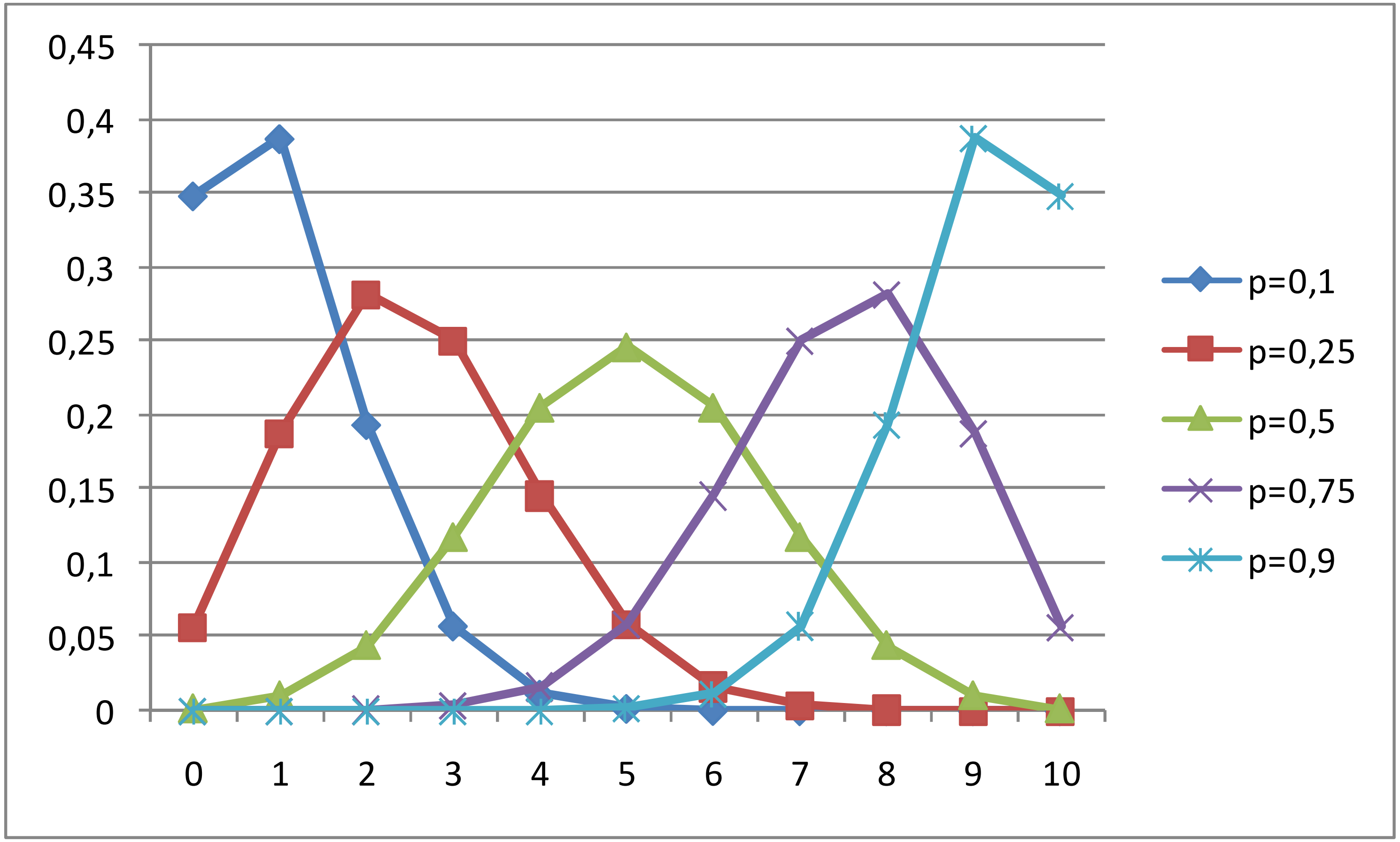
$$\mathbb{P}(\{(a_1,\, a_2,\, \ldots,\, a_n)\}) = p^{\sum_{i=1}^{n} a_i}(1-p)^{n-\sum_{i=1}^{n} a_n}$$

- probability of exactly *k* successes in *n* trials

$$\binom{n}{k} p^k (1-p)^{n-k}$$

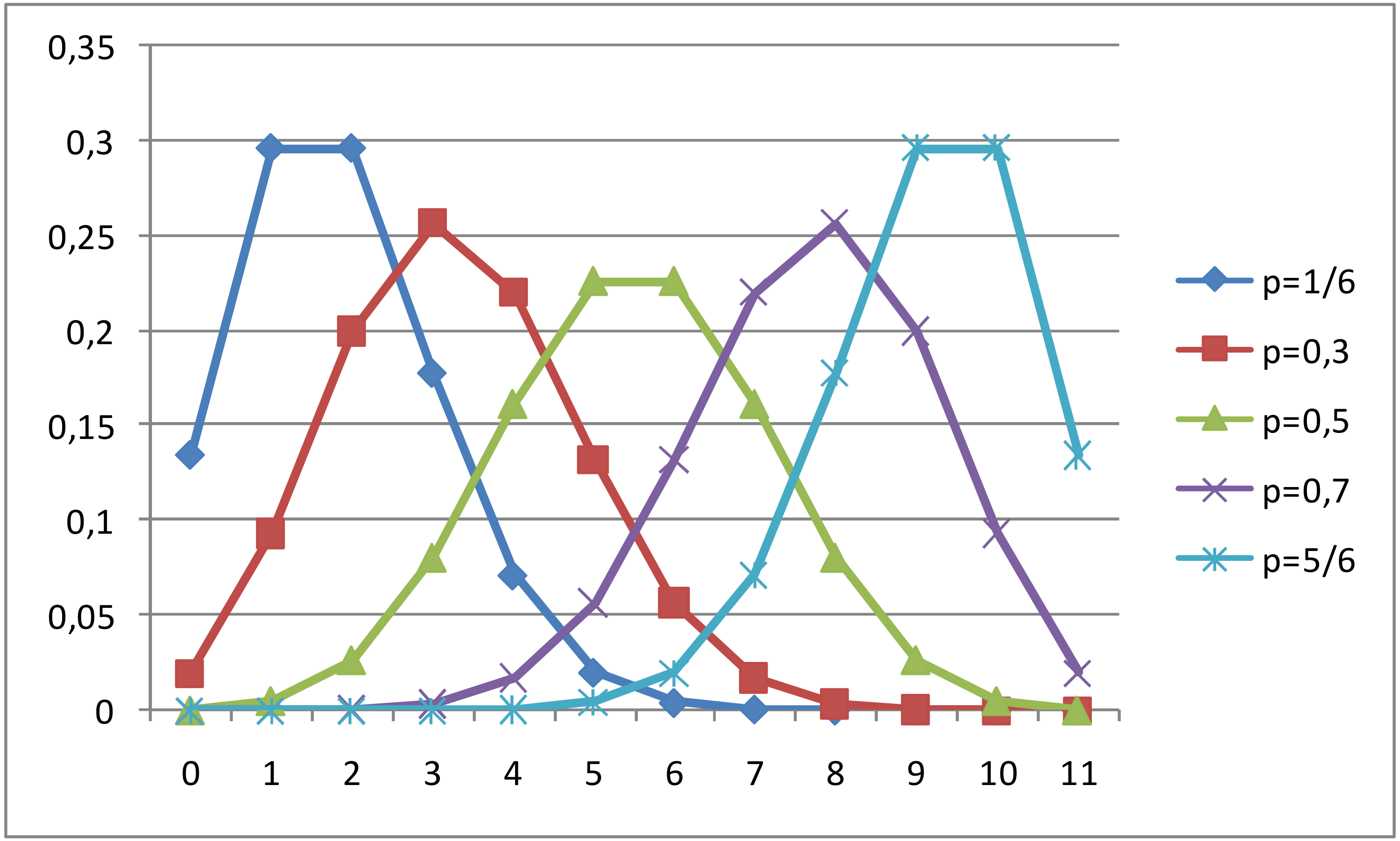
Bernoulli Process – cont. (2)

n=10



Bernoulli Process – cont. (3)

n=11



Bernoulli Process – cont. (4)

4. Examples
   - coin flip
   - die roll
5. The most probable number of successes
6. Infinite sequence of heads

Poisson Theorem

1. Poisson Theorem

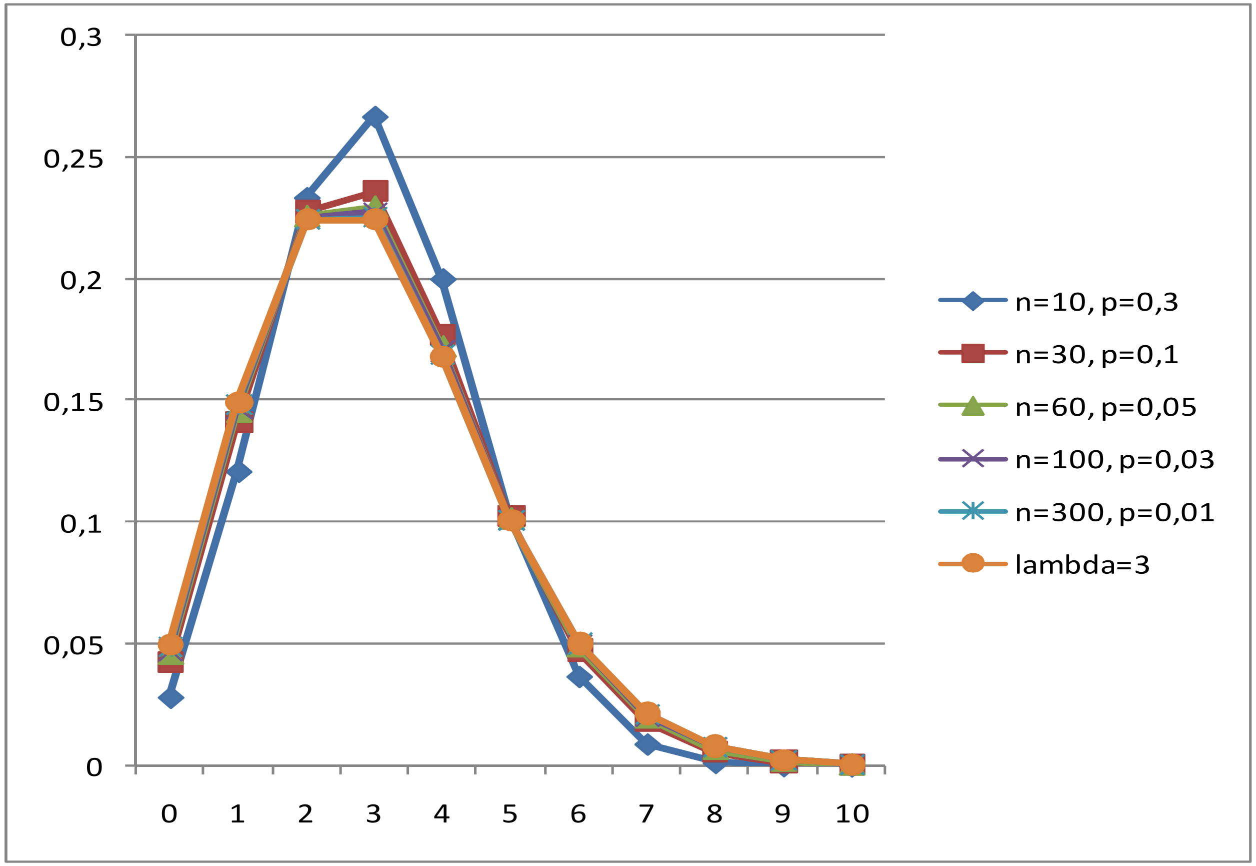
   If $p_n \in [0,1]$, $\lim_{n\to\infty} np_n = \lambda > 0$,
   then for $k = 0, 1, 2, \ldots,$
   we have that
   $\lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$

2. Assessment of approximation error

   Let S_n *denote the number of successes in a Bernoulli process with* n *trials and a probability of success in a single trial equal to* p, *and let* $\lambda = np$. *For any* $A \subset \{0, 1, 2, \ldots\}$, *we have*
   $\left|\mathbb{P}(S_n \in A) - \sum_{k\in A} \frac{\lambda^k}{k!} e^{-\lambda}\right| \leqslant \frac{\lambda^2}{n}.$

Poisson Theorem – cont.

The Poisson and Bernoulli processes



Poisson Theorem – cont. (2)

3. Examples of good and not so good approximations