

Probability Calculus

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INTRODUCTION TO PROBABILITY CALCULUS

Some technicalities

- Contact: ajanicka@wne.uw.edu.pl
- Office hours: Tue, 9:45 AM, FoES room A006
- web page (materials):
www.wne.uw.edu.pl/azylicz
- Readings:
 - **Charles M. Grinstead and J. Laurie Snell, Introduction to Probability, available online**
 - **Sheldon M. Ross, Introduction to Probability Models, available in the FoES library and online**
 - **Wackerly, D., Mendenhall, W., & Scheaffer, R. Mathematical statistics with applications, available in the FoES library**



Assessment

1. Presence during lectures – recommended; presence during classes – mandatory
2. Class assessment: Necessary condition: at least 6 out of 7 short tests passed + activity during classes
3. Homework
4. Lecture assessment: Two tests, one on December 3rd, 6:45 PM Aud Max (40pts), one at the end of the semester (50pts) + homework (10pts)
5. Evaluation 1st period: Test1 + Test2 + homework
6. Evaluation 2nd period: $\max \{ \text{Test1} + \text{Test2b} + \text{homework}, 9/5 \text{ Test2b} + \text{homework} \}$



What to expect

- Lecture notes (web page)
- Problems to solve during classes (web page)
- Homework (web page)
- Preparation for classes: previous lecture material + previous classes



Probability Calculus 1st term exam results and homework

HW pts	Average of exam points
0	16.16667
1	8.25
2	13.625
3	14.5
4	15.1
5	15.25
6	16.90909
7	20.21552
8	20.4
9	21.70089
10	24.77218

....

Is what we see a coincidence?

Is there a relationship between the number of points obtained during the exam and the number of homework points?

Examples of other questions:

What is the probability that a student will pass the exam if the number of homework points is equal to 10? 0?

What is the chance that everybody will pass?

What does the distribution of points depend on?



Thematic scope of course

- ❑ Some basics and „classics”
- ❑ „Contemporary” probability
- ❑ Reality description – random variables.
Crucial in statistics and econometrics
- ❑ Limit theorems – crucial as above, very important in practice (e.g. insurance)



Plan for today

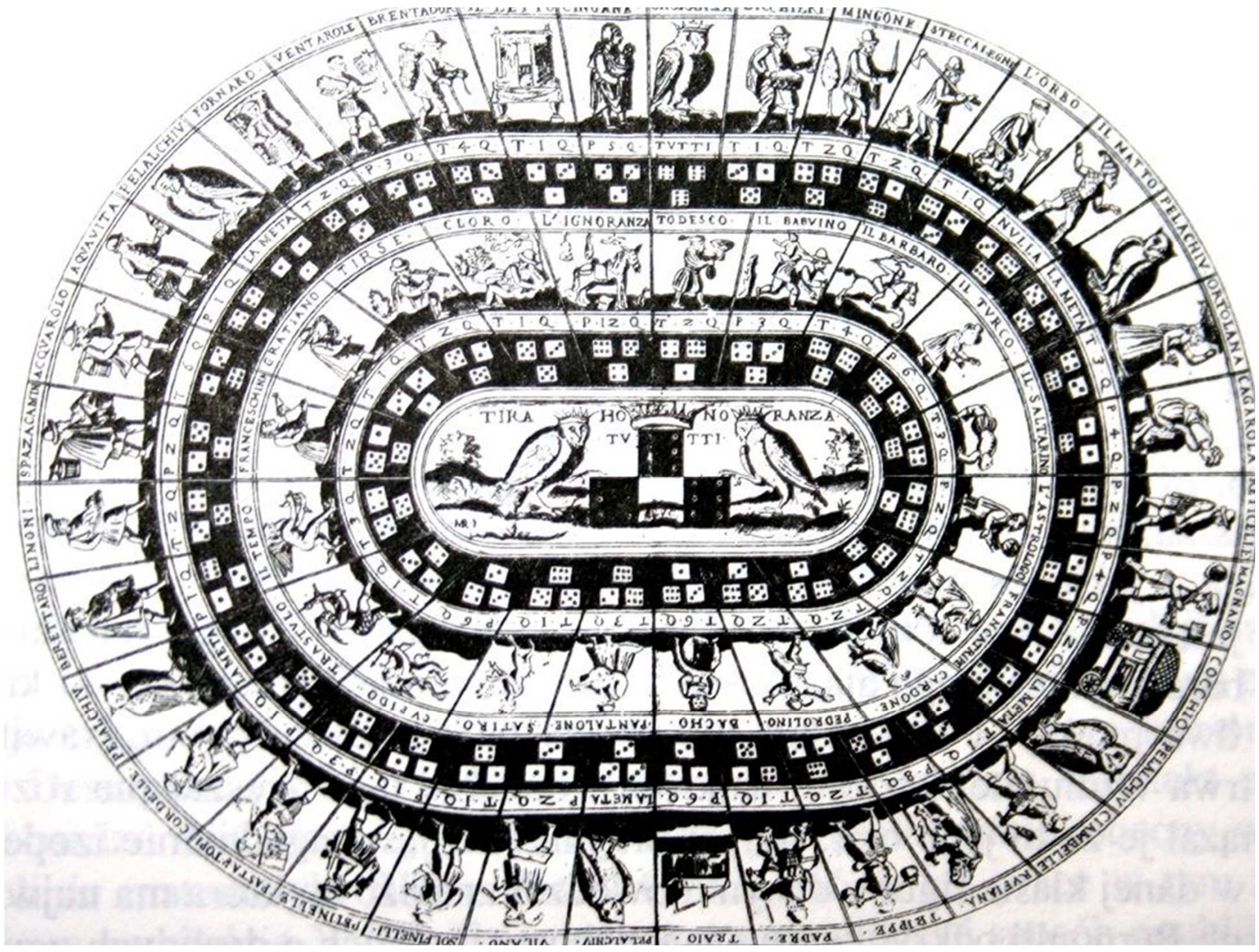
1. Historical perspective
2. Basic definitions and notations, examples
3. *σ -algebras*
4. Probability intuitively and **Kolmogorov axioms**,
5. Examples
6. Basic properties of probability



1. Historical perspective

- Motivation:
 - gambling
 - statistics of births and deaths
 - insurance of transports
 - „Paradoxes”
 - First mathematical publications without errors: Bernoulli, 1752
 - „Contemporary probability”: Kolmogorov axioms, 1933
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Gerolamo Cardano, De Ludo Aleae (Book on Games of Chance), 1564

"If it is necessary for someone that he should throw at least twice, then you know that the throws favorable for it are 91 in number, and the remainder is 125; so we multiplying each of these numbers by itself and get to 8,281 and 15,625, and the odds are about 2 to 1."

*"**This reasoning seems to be false...** for example, the chance of getting one of any three chosen faces in one cast of one dice is equal to the chance of getting one of the other three, but according to this reasoning there would be an even chance of getting a chosen face each time in two casts, and thus in three, and four, **which is***

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2. Basic definitions and notations

Elementary event: ω

Sample space : Ω

Event: $A, B, \text{ etc.}$

Special events, operations:

$\Omega, \emptyset, A', A \cup B, A \cap B, A \setminus B, A \subseteq B$



2. Examples

1. Coin toss
2. Dice rolling
3. Rolling of a pair of dice – sum of points
4. Draw of 13 cards out of 52 – with and without order
5. Coin toss until first „heads”
6. Needle on a table



3. σ -algebra

Defines the sets that we can measure (calculate probability). In most simple cases: **we don't need to worry about it.**

Definition of a σ -algebra \mathcal{F} of subsets of Ω

A family \mathcal{F} of subsets of Ω is called a σ -algebra, if

- (i) $\emptyset \in \mathcal{F}$,*
- (ii) $A \in \mathcal{F} \Rightarrow A' \in \mathcal{F}$,*
- (iii) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.*



4. Probability intuitively – frequencies

$$\rho_n(A) = \frac{\text{number of occurrences of } A}{n}$$

- Calculating frequencies
- Properties of frequencies
- Limit =?



4. Probability formally - Kolmogorov Axioms

□ For a given (Ω, \mathcal{F}) we define probability as a function satisfying 3 conditions

- (i) $0 \leq \mathbb{P}(A) \leq 1$,
- (ii) $\mathbb{P}(\Omega) = 1$,
- (iii) if $A_1, A_2, \dots \in \mathcal{F}$ are pairwise disjoint

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

□ Probability space $(\Omega, \mathcal{F}, \mathbb{P})$



5. Examples

1. Symmetric coin toss, asymmetric coin toss
2. Dice rolling
3. **Classic scheme (simple probability)**
4. Drawing cards
5. Countable sample spaces
6. Geometric probability



5. Basic properties of probability

□ Theorem 1 (arithmetics)

Theorem 1. Let $A, B, A_1, A_2, \dots \in \mathcal{F}$. Then

- (i) $\mathbb{P}(\emptyset) = 0$,
- (ii) If A_1, A_2, \dots, A_n are pairwise disjoint, then $\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i)$.
- (iii) $\mathbb{P}(A') = 1 - \mathbb{P}(A)$.
- (iv) If $A \subseteq B$, then $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$.
- (v) If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- (vi) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
- (vii) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.



5. Examples – cont.

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5. Basic properties of probability – cont.

□ Theorem 2 (inclusion-exclusion principle)

If $A_1, A_2, \dots, A_n \in \mathcal{F}$, then

$$\begin{aligned} \mathbb{P}(A_1 \cup A_2 \cup \dots \cup A_n) &= \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \dots \\ &\quad + (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) \end{aligned}$$



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