Probability Calculus Final Exam
February 8th, 2012

- The solution of a problem must include all calculations and all the steps of the reasoning, recall all theorems and formulae used, etc. A solution consisting of the final answer only will receive 0 pts.
- It is prohibited to use any notes, books, tables or calculators. Mobile phones must be switched off at all times.
- Total exam time: 150 minutes.

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1. A contemporary painter creates his artwork in the following way: he randomly chooses a point (X, Y) from a white square canvas $[0,1] \times [0,1]$, and then paints the lower left corner (the area underneath and to the left of (X, Y)) green, and the upper right corner (the area above and to the right of (X, Y)) orange. What will be, on average, the area of the colored fraction of the canvas? (*4 pts*) Find the distribution of the random variable $X - Y$ (*4 pts*).

2. Let (X, Y) be a two-dimensional random variable with density $g(x, y) = (x+y)1_{\{0 \leqslant x \leqslant 1\}}1_{\{0 \leqslant y \leqslant 1\}}$. Find the marginal distributions of X and Y (*4 pts*). Calculate the covariance of X and Y (*4 pts*). Are X and Y independent? Justify the answer. (*2 pts*).

3. Let (X, Y) be a two-dimensional random variable with density $g(x, y) = \frac{e^{-y}}{y}1_{\{0<x<y\}}$. Calculate $\mathbb{E}(X \mid Y)$ (*4 pts*), $\mathbb{E}(X^2 \sin(Y) \mid Y)$ (*4 pts*), and $\mathbb{E}X$ (*2 pts*).

4. The number of main prize Lotto winners (i.e. all six numbers matched) in a single lottery is a random variable with a Poisson distribution with parameter $\lambda = 1/4$. Let us assume that every holder of a winning ticket has a 90% chance of claiming his prize. Let X denote the number of main prize winners, and Y — the number of claimed first prizes. Describe the conditional distribution $Y|X = x$ (*2 pts*). Calculate $\mathbb{E}(Y|X)$ (*3 pts*) and $\mathbb{E}Y$ (*2 pts*).

5. We carry out a sequence of coin tosses with a symmetric coin. We define:
   (a) random variables X_n in the following way: $X_n = 0$, if the n-th result was a head, and $X_n = 2$ otherwise. Verify whether the sequence $\frac{X_1+\ldots+X_n}{2n+1}$ converges almost surely and find the limit if it does. (*4 pts*)
   (b) random variables Y_n in the following way: $Y_n = 0$, if there were only heads in the first n tosses, and $Y_n = 1$ otherwise. Verify whether the sequence Y_n converges in probability and find the limit if it does. (*4 pts*)

6. Assume that, on average, every other supermarket customer economizes, and the rest are high-spending. Approximate the probability that among a group of 100 (independent) customers, at most 55 will be high-spending (*4 pts*). Let us further assume that an economical customer spends on average \$100, with a standard deviation equal to \$30; whereas a high-spending customer – on average \$300, with a standard deviation of \$40. Approximate the probability that 200 customers (out of which 100 are economical, and 100 high-spending) will spend at least \$38 thousand (*5 pts*).

7. Jack and Jill play the following game: they toss a symmetric coin until the sequence HEAD-HEAD-TAIL or TAIL-TAIL-TAIL appear. If the former appears first, Jack wins; if the latter – Jill triumphs. What is the chance that after at most five tosses, Jack will win? (*3 pts*) What is the chance that Jack will win (in general)? (*5 pts*)

$\Phi(1) \approx 0,841$, $\Phi(1,5) \approx 0,933$, $\Phi(2) \approx 0,977$, $\Phi(2,5) \approx 0,994$, $\Phi(3) \approx 0,9987$, $\Phi(4) \approx 0,99997$