Probability Calculus 2018/2019
Problem set 14

1. Consider a time-homogeneous Markov chain $(X_n)_{n\geqslant 0}$, such that

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

   a) Calculate $p_{12}(2)$.
   b) Assuming $X_0 = 1$ (with probability 1), find the probability that X_n will reach state 2 before it reaches state 4.
   c) Find m_{32}.
   d) Is the chain periodic? Irreducible?
   e) Find the stationary distribution.
   f) Approximate the probability that $X_{10000} = 1$.
   g) Find the mean recurrence time for state 1.

2. We roll a die until we obtain 16 or 66. What is the probability that we will obtain 16 first?

3. A frog jumps from stone to stone. There are five stones, forming a regular pentagon (ABCDE). Once on a stone, the frog chooses randomly (independently from previous choices) either the stone to the left, or to the right, with probabilities $\frac{1}{2}$ and $\frac{1}{2}$. The frog starts from stone A.

   a) Calculate the probability that the frog will return to stone A before it reaches stone C.
   b) Calculate the mean recurrence time for a stone and compare with the stationary distribution.

4. We toss a coin until we obtain three heads in a row. Find the mean number of tosses.

5. Smith is in jail and has 1 dollar; he can get out on bail if he had 4 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6. What is the better strategy to get out on bail (win 4 dollars before losing all of his money):

   a) timid strategy: to bet 1 dollar each time?
   b) bold strategy: to bet as much as possible each time?

Some additional simple problems you should be able to solve on your own:

A. A time-homogenous Markov chain $(X_n)_{n\geqslant 0}$ over the space $E = \{1,2,3\}$ has the following transition matrix:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 0 \end{pmatrix}.$$

a) What is the probability that starting from state 2, after two steps the Markov chain will again be in state 2?

b) Assuming $X_0 = 1$ a.s. calculate the probability that the Markov chain will return to state 1 before it reaches state 3.

c) Assuming $X_0 = 3$ a.s. calculate the mean expected number of steps for reaching state 1.

d) Find the stationary distribution. Is the Markov chain irreducible? Periodic?

e) Approximate the probability that $X_{10000} = 1$.

f) Calculate the mean recurrence times for the states and compare with the stationary distribution.