Probability Calculus 2018/2019
Problem set 11

1. Employment levels in different sewing factories in Bangladesh were compared. The number of needlewomen employed in particular units was a random variable with a uniform distribution over the interval $[100, 300]$. The wage of a needlewoman depends on the institution, and has an average of 2400 with a standard deviation of $400\sqrt{3}$. The correlation coefficient of the employment level and the wage level in the units compared was equal to 0.6. Find the best linear approximation of the relationship between wages and the employment level in a given factory.

2. We roll a die twice. Let X and Y denote the numbers obtained in the first and second roll, respectively. Calculate $\mathbb{E}(Y|X)$, $\mathbb{E}(X+Y|X)$ and $\mathbb{E}(X|X+Y)$.

3. From the set $\{1, 2, \ldots, 10\}$ we randomly draw, without replacement, two numbers. Let X be the smaller and Y the larger of the two values. Calculate $\mathbb{E}(X|Y)$ and $\mathbb{E}(XY+X|X)$.

4. The monthly energy usage in a plant has a uniform distribution over $[200, 250]$. For a given usage level λ, the amount of CO_2 emissions has an exponential distribution with parameter $5 - \lambda/100$. Find the (unconditional) density of the level of emissions.

5. Let (X, Y) be a random vector with a uniform distribution over a triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Calculate $\mathbb{E}(Y|X)$, $\mathbb{E}(XY^2 + 3X^2Y - 1|X)$ and $\mathbb{P}(Y \leqslant \frac{1}{2}|X)$.

6. We roll a die once, and then again as many times as there were points during the first roll. Let X denote the total sum of points obtained during the experiment (including the first roll). Find $\mathbb{E}X$.

Some additional simple problems you should be able to solve on your own:

Theory (you should know going into class 11)

1. What is the definition of a conditional expectation for a discrete random variable?
2. What is a conditional density function? What is the definition of a conditional expectation for a continuous random variable?

Problems (you should know how to solve after class 11)

3. A transmitter sends signal X. A receiver picks up signal $Y = aX + Z$, where $a > 0$ is the amplification factor, and Z is interference. X and Z are independent random variables, such that $\mathbb{E}X = m$, $\operatorname{Var} X = 1$, $\mathbb{E}Z = 0$ and $\operatorname{Var} Z = \sigma^2$. Find the correlation coefficient of X and Y and the linear regression of X and Y.
4. Knowing that $\mathbb{P}(Y = 1|X = 5) = 1/3$ and $\mathbb{P}(Y = 5|X = 5) = 2/3$, find $\mathbb{E}(Y|X = 5)$ and $\mathbb{E}(XY^2|X = 5)$.
5. There are two white balls, with numbers 1 and 2, and three black balls, with numbers 1, 2 and 3, in a box. Two balls were drawn from the box without replacement. Let X denote the maximum number obtained, and Y denote the number of white balls drawn. Find $\mathbb{E}(Y|X)$ and $\mathbb{E}(X|Y)$.
6. A coin was tossed three times. Let X denote the number of heads and

$$Y = \begin{cases} 1 & \text{if the last toss was heads,} \\ 0 & \text{otherwise.} \end{cases}$$

   Find $\mathbb{E}(X|Y)$ and $\mathbb{E}(XY|X)$.
7. Let (X, Y) be a random vector from a uniform distribution over a triangle with vertices $(2, 0)$, $(0, 1)$ and $(-1, 0)$. Calculate $\mathbb{E}(X|Y)$ and $\mathbb{E}(X^2 + XY|Y)$.
8. Let (X, Y) be a random vector with density

$$g(x, y) = (x + y)1_{\{0 \leqslant x \leqslant 1,\, 0 \leqslant y \leqslant 1\}}.$$

   Find $\mathbb{E}(X|Y)$ and $\mathbb{E}(\sin X + Y|Y)$.