

Probability Calculus

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**INDEPENDENCE OF RV
LINEAR REGRESSION**

Plan for Today

1. Independence
2. Multidimensional Normal RV
3. Linear regression



Independent RV

1. Definition of independence

Variables $X_1, \dots, X_n : \Omega \rightarrow \mathbb{R}$ are **independent**, if for any sequence of Borel sets B_1, B_2, \dots, B_n , we have

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2, \dots, X_n \in B_n) = \mathbb{P}(X_1 \in B_1) \cdot \mathbb{P}(X_2 \in B_2) \cdot \dots \cdot \mathbb{P}(X_n \in B_n).$$

2. Independence of discrete RV

Let X_1, X_2, \dots, X_n be discrete random variables with supports S_{X_i} , respectively. In this case, X_1, X_2, \dots, X_n are independent if and only if for any sequence x_1, x_2, \dots, x_n such that $x_i \in S_{X_i}$, $i = 1, 2, \dots, n$, we have

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdot \mathbb{P}(X_2 = x_2) \cdot \dots \cdot \mathbb{P}(X_n = x_n).$$



Independent RV – cont.

3. Example

4. Independence of continuous RV

Let $X_1, X_2, \dots, X_n: \Omega \rightarrow \mathbb{R}$ be continuous random variables with probability densities g_1, g_2, \dots, g_n , respectively. In this case, X_1, X_2, \dots, X_n are independent if and only if $g: \mathbb{R}^n \rightarrow [0, \infty)$, defined as

$$g(x_1, x_2, \dots, x_n) = g_1(x_1) \cdot g_2(x_2) \cdot \dots \cdot g_n(x_n),$$

is a density function of the distribution $\mu_{(X_1, X_2, \dots, X_n)}$.

5. Examples

- uniform distribution on square
- uniform distribution on circle



Independent RV – cont. (2)

6. Transformations of RV

Let $X_{1,1}, X_{1,2}, \dots, X_{1,k_1}, X_{2,1}, X_{2,2}, \dots, X_{2,k_2}, \dots, X_{n,1}, X_{n,2}, \dots, X_{n,k_n}$ be independent random variables, and $\varphi_i : \mathbb{R}^{k_i} \rightarrow \mathbb{R}$, $i = 1, 2, \dots, n$ be Borel functions. We then have that the variables

$$Y_1 = \varphi_1(X_{1,1}, X_{1,2}, \dots, X_{1,k_1}),$$

$$Y_2 = \varphi_2(X_{2,1}, X_{2,2}, \dots, X_{2,k_2}),$$

...

$$Y_n = \varphi_n(X_{n,1}, X_{n,2}, \dots, X_{n,k_n})$$

are independent.



Properties of independent RV

2. Expected value of product

Let X_1, X_2, \dots, X_n be independent random variables with expected values. Then, the variable $X = X_1 \cdot X_2 \cdot \dots \cdot X_n$ also has an expected value, and we have

$$\mathbb{E}X = \mathbb{E}(X_1 \cdot X_2 \cdot \dots \cdot X_n) = \mathbb{E}X_1 \cdot \mathbb{E}X_2 \cdot \dots \cdot \mathbb{E}X_n.$$

3. Example

4. Covariance of independent RV

Let X and Y be independent random variables, such that $\mathbb{E}|XY| < \infty$. We then have $\text{Cov}(X, Y) = 0$.

5. Non-correlation



Properties of independent RV – cont.

6. One-way implication only!

independence \Rightarrow non-correlation but

\Leftarrow IS NOT TRUE!

7. Example – uniform distribution on circle

8. Sum of variances

Let X_1, X_2, \dots, X_n be ~~independent~~ random variables with finite variances. Then, the variable $X = X_1 + X_2 + \dots + X_n$ also has a finite variance, and we have

$$\begin{aligned} \text{Var}X &= \text{Var}(X_1 + X_2 + \dots + X_n) \\ &= \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n) \\ &\quad + 2 \sum_{i < j} \text{Cov}(X_i, X_j). \end{aligned}$$



Properties of independent RV – cont. (2)

9. Example – sum of points on dice

10. Convolution of density functions

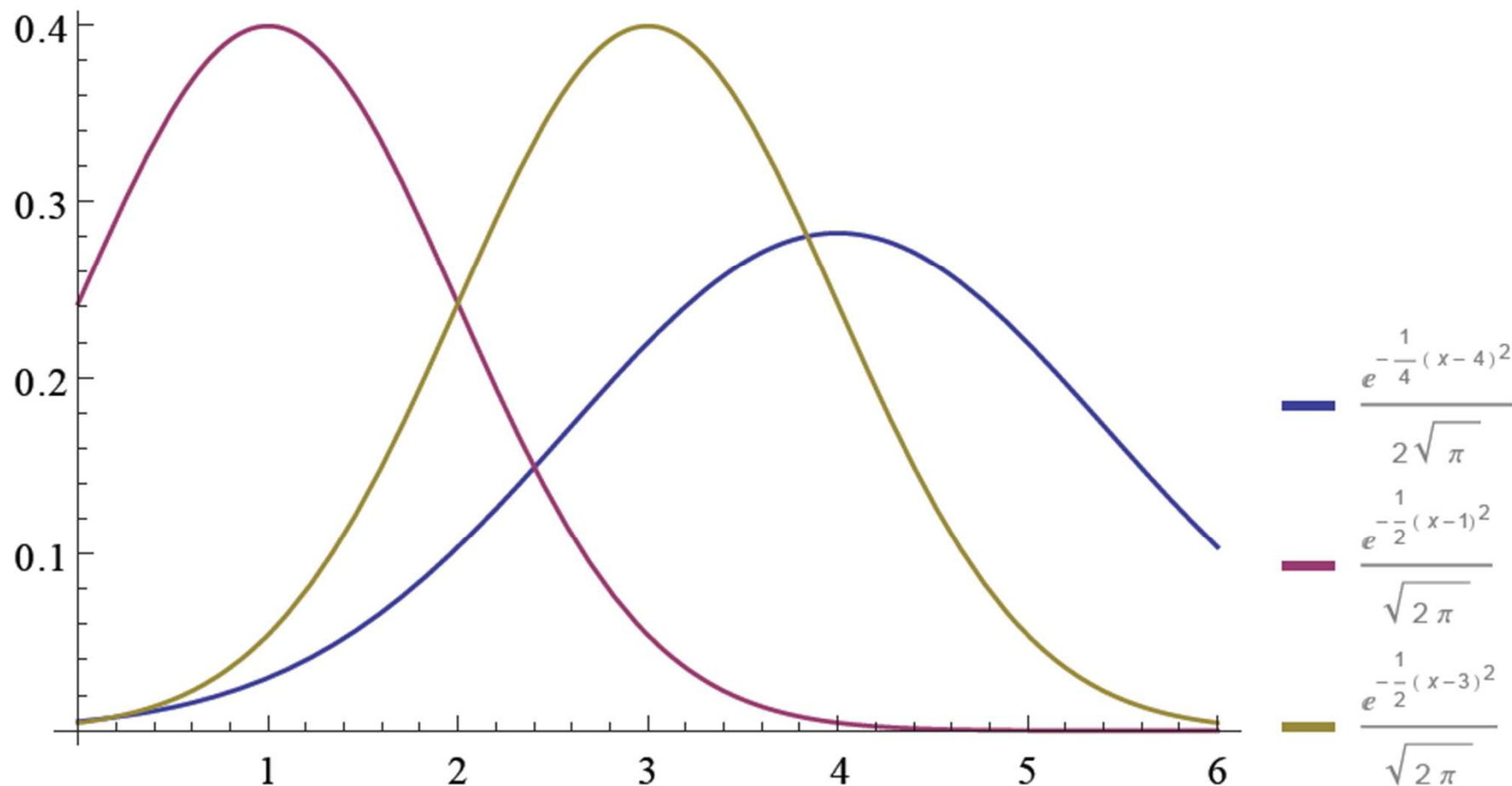
*Let X and Y be independent random variables with densities g_X and g_Y , respectively. Then, the density of the variable $X + Y$ may be presented as a **convolution** of densities g_X and g_Y :*

$$\begin{aligned}g_{X+Y}(t) &= g_X * g_Y(t) \\ &= \int_{\mathbb{R}} g_X(x)g_Y(t - x)dx = \int_{\mathbb{R}} g_X(t - y)g_Y(y)dy\end{aligned}$$

11. Example



Convolution of densities – example



Multidimensional Normal RV

1. Definition

Let $m = (m_1, m_2, \dots, m_n)$ be a vector in \mathbb{R}^n and let A be a positive definite $n \times n$ matrix (i.e. such that $x^t A x > 0$ for any nonzero vector $x \in \mathbb{R}^n$). A distribution with density

$$g(x) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} \exp\left(-\frac{(x-m)^t A (x-m)}{2}\right), \quad x \in \mathbb{R}^n$$

is a **normal** distribution with mean m and a covariance matrix $Q = A^{-1}$.

2. Affine transformations of normal RV



Two-dimensional normal RV

3. Two-dimensional normal RV with mean

$m = (m_1, m_2)$ and a covariance matrix Q

$$g(x, y) = \frac{\sqrt{a_{11}a_{22} - a_{12}^2}}{2\pi} \cdot \exp\left(-\frac{1}{2}(a_{11}(x - m_1)^2 + 2a_{12}(x - m_1)(y - m_2) + a_{22}(y - m_2)^2)\right)$$

$\sqrt{\det A}$

$(x - m)^t A (x - m)$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = Q^{-1}$$



Condition of independence of normal RV

4. Theorem

Let $X = (X_1, X_2, \dots, X_n)$ be a normal variable, and let X_1, X_2, \dots, X_n be uncorrelated. Then, X_1, X_2, \dots, X_n are independent.



Linear regression

1. Best (in terms of average square deviation) **linear** approximation of variable Y with variable X , i.e. $aX+b$:
minimizes

$$f(a, b) = \mathbb{E}(Y - aX - b)^2$$

solution:

$$a = \frac{\text{Cov}(X, Y)}{\text{Var}X} \quad b = \mathbb{E}Y - \frac{\text{Cov}(X, Y)}{\text{Var}X} \mathbb{E}X$$



$$a = \rho_{X,Y} \cdot \frac{\sigma_Y}{\sigma_X}$$