Probability Calculus 2017/2018
Problem set 9

1. A die is rolled twice. Let X denote the number of sixes obtained, and Y – the number of ones.
   a) Find the support of (X, Y)
   b) Calculate the covariance of X and Y
   c) Find the probability that $X > Y$.

2. A random vector (X, Y) has a distribution given by
$$\mathbb{P}\Big((X,Y) = (k,l) \Big) = \frac{4kl}{n^2(n+1)^2}, \qquad k,\, l = 1, 2, \ldots, n.$$
   a) Find $\mathbb{P}(X + Y = n + 1)$.
   b) Find the marginal distributions of X and Y.
   c) Calculate $\operatorname{Cov}(X, Y)$.

3. A random vector (X, Y) has a density $g(x, y) = Cx\mathbb{1}_{\{0 \leqslant x \leqslant 1,\, 0 \leqslant y \leqslant 1\}}$.
   a) Calculate C
   b) Calculate $\mathbb{P}(X + Y < 1)$ and $\mathbb{P}(Y \leqslant 1/2)$.
   c) Find the marginal distributions of X and Y.
   e) Calculate $\operatorname{Cov}(X, Y)$.

4. The covariance matrix of random vector (X, Y) is equal to
$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$
   Calculate the correlation coefficient of $X + 3Y$, $2X - Y$.

5. Let (X, Y) be a random vector with density
$$g(x, y) = xe^{-y}\mathbb{1}_{\{0 \leqslant x \leqslant y\}}.$$
   a) Find the marginal distribution of X.
   b) Calculate $\mathbb{E}e^{Y/2}$.
   c) Calculate the CDF of (X, Y) at $(1, 1)$.
   d) Find the distribution of the variable Y/X.

Some additional problems

Theory (you should know going into class 9)

1. What is the CDF of a two-dimensional random vector (X, Y)? What are the marginal distributions?

2. Define the covariance and the correlation coefficient of variables X, Y.

3. Define the covariance matrix of variable (X, Y).

Problems (you should know how to solve after class 9)

4. A symmetric coin was tossed three times. Let X denote the number of heads in the last toss, and Y - the overall number of heads. Find $\mathbb{P}(X = Y)$ and $\mathrm{Cov}(X, Y)$.

5. Let (X, Y) be a random vector with density $g(x, y) = Cxy\mathbb{1}_{\{(x,y):0 \leqslant x \leqslant y \leqslant 1\}}$. Find C and $\mathbb{P}(X \geqslant 1/2)$.

6. Let (X, Y) be a random vector with density $g(x, y) = \mathbb{1}_{\{0 \leqslant y \leqslant 1-|x|\}}$.

a) Find the CDF of (X, Y) at point $\left(1, \frac{1}{2}\right)$.

b) Find the (marginal) densities of X and Y.

c) Calculate $\mathrm{Cov}(X, Y)$.

d) Find the distribution of $Y - X$.

7. We roll a regular die until a six appears. Let X denote the overall number of tosses, and Y - the number of fives.

a) Find the distribution of (X, Y).