

# Probability Calculus

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**DISTRIBUTIONS OF RANDOM VARIABLES – CONT.:  
CUMULATIVE DISTRIBUTION FUNCTION, EXPECTED VALUE –  
INTRO**

# Plan for today

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- Cumulative Distribution Functions
- Transformations of random variables
- Quantiles
- Expected value for discrete random variables



# Random variables – the CDF

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## 1. The definition of a CDF

**The Cumulative distribution function**

*of a random variable  $X : \Omega \rightarrow \mathbb{R}$*

*is a function  $F_X : \mathbb{R} \rightarrow [0, 1]$ , such that*

$$F_X(t) = \mathbb{P}(X \leq t).$$

depends on the distribution only!  
→ CDF of distribution



# Random variables – the CDF

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## 2. Examples of CDFs

- Dirac delta
- Two-point distribution – discrete distribution
- Exponential distribution
- Normal distribution – no simple form...



# CDFs

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## 3. Properties of the CDF

*The cumulative distribution function  $F_X$  of a random variable  $X$  has the following properties:*

- (i)  $F_X$  is nondecreasing,*
- (ii)  $\lim_{t \rightarrow \infty} F_X(t) = 1$  and  $\lim_{t \rightarrow -\infty} F_X(t) = 0$ ,*
- (iii)  $F_X$  is right-continuous.*

## 4. CDF $\rightarrow$ distribution

*For any function  $F : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the conditions (i)-(iii) above, there exists a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a random variable  $X : \Omega \rightarrow \mathbb{R}$  such that  $F$  is the CDF of  $X$ . Furthermore, the distribution of  $X$*

* is determined unequivocally.*

## CDFs – cont.

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5. A CDF of a discrete distribution

6. Further properties of the CDF:

*If  $F_X$  is a cumulative distribution function of a random variable  $X$ , then for all  $t \in \mathbb{R}$  we have  $F_X(t-) = \mathbb{P}(X < t)$  and  $F_X(t) - F_X(t-) = \mathbb{P}(X = t)$ . In particular, if  $F_X$  is continuous at point  $t$ , then  $\mathbb{P}(X = t) = 0$ .*



## CDFs – cont (2)

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### 7. CDF $\rightarrow$ density

*Let  $F$  be the CDF of a random variable  $X$ .*

- 1. If  $F$  is not continuous, then  $X$  does not have a continuous distribution (does not have a density function).*
- 2. Assume  $F$  is continuous. If  $F$  is differentiable apart from a finite set of points, then the function*

$$g(t) = \begin{cases} F'(t) & \text{if } F'(t) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

*is a density function for  $X$ .*

### 8. Examples

- uniform distribution
- distribution that is neither discrete nor continuous



# Transformation of random variables

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## 9. Well-behaved transformations of continuous variables

*Assume  $X$  is a random variable with density  $f$ .*

*If the values of  $X$  fall within the interval  $(a, b)$*

*(with probability 1), and  $\varphi : (a, b) \rightarrow \mathbb{R}$  is  $C^1$*

*and  $\varphi'(x) \neq 0$  for  $x \in (a, b)$ , then*

*$Y = \varphi(X)$  is continuous with a density function*

$$g(y) = f(h(y))|h'(y)|1_{\varphi((a,b))}(y),$$

*where  $h(s) = \varphi^{-1}(s)$ .*

## 10. Example

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# Quantiles

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## 1. Definition

*Let  $X$  be a random variable and  $p \in [0, 1]$ .*

*A quantile of rank  $p$  of the variable  $X$  is any value  $x_p$ , such that*

$$\mathbb{P}(X \leq x_p) \geq p \text{ and}$$

$$\mathbb{P}(X \geq x_p) \geq 1 - p.$$

## 2. Examples

- continuous distribution ( $N(0,1)$ )
- discrete distribution



# Expected value – discrete RV

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1. Motivation & intuition
2. Definition of expected value for discrete RV

*Let  $X$  be a random variable with a discrete distribution, concentrated on  $S \subset \mathbb{R}$ , and let  $p_x = \mathbb{P}(X = x)$  for  $x \in S$ . We will say that the expected value of  $X$  is finite if  $\sum_{x \in S} |x|p_x < \infty$ . Then we can define this **expected value** of  $X$  as  $\mathbb{E}X = \sum_{x \in S} xp_x$ .*

mean value, depends on the distribution only for a finite set  $S$ , the  $\mathbb{E}X$  always exists

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## Expected value – discrete RV. cont.

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### 3. Examples of calculations

- single-valued RV
- die roll
- Binomial distribution  $(n,p)$
- variables without EX:
  - series does not converge at all
  - series does not converge absolutely





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