Probability Calculus 2018/2019
Problem set 5+6

This set is for groups which did not have classes during the week of October, 29th - November, 2nd

1. Let F denote the CDF of a random variable X, defined by:

$$F(t) = \begin{cases} 0 & \text{if } t < -2, \\ \frac{1}{3} & \text{if } t \in [-2, 0), \\ \frac{1}{3}t + 1/2 & \text{if } t \in [0, 1), \\ \frac{5}{6} & \text{if } t \in [1, 5), \\ 1 & \text{if } t \geqslant 5. \end{cases}$$

   Calculate $\mathbb{P}(X \in (3,7))$, $\mathbb{P}(X \in [-2,-1])$, $\mathbb{P}(X \in [-2,-1))$, $\mathbb{P}(X \in (-2,-1))$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 2)$. Is the distribution of X discrete? Is the distribution of X continuous? Find the quantile of rank $\rho = 1/4$.

2. Let X be a random variable with density $g(x) = \frac{3}{8}x^2 1_{(0,2)}(x)$. Find the distributions of a) $\max\{X, 1\}$, b) X^{-2}. Are these distributions continuous? If yes, calculate the density.

3. Let X be a standard normal variable. Find the density of $Y = e^X$ and $Y = X^2$.

4. Find the quantile of rank $p = 5/16$ for a) an exponential distribution with parameter λ, b) a Binomial distribution with parameters 4 and $\frac{1}{2}$.

5. Let X be a random variable such that

$$\mathbb{P}(X = -1) = \mathbb{P}(X = 0) = \frac{1}{4}, \quad \mathbb{P}(X = 3) = \frac{1}{3}, \quad \mathbb{P}(X = 5) = \frac{1}{6}.$$

   Calculate $\mathbb{E}X$ and $\mathbb{E}(2X - 1)$.

6. Consider the following game: we toss a symmetric coin until heads appear. If heads appear in the n-th toss, we win $(1.5)^n$ dollars. What is a reasonable price for participation in this game? And if the gain for tails in the n-th toss was 2^n dollars?

Some additional problems

Theory (you should know after the fifth & sixth lecture and before the fifth class):

1. What is the CDF of a probability distribution? What are the properties that a function must fulfill in order to be a CDF? How do we determine the type of distribution (discrete/continuous) from the CDF?

2. What is a quantile of rank p?

3. Define the expected value of a discrete random variable X.

4. Describe the properties of the expected value operator.

Problems (you should know how to solve after class 5)

1. Let X be a random variable with density $g(x) = \frac{1}{2}\sin x\, 1_{[0,\pi]}(x)$. Show that $\pi - X$ has the same distribution as X.

2. Let X be a random variable from a binomial distribution $B(n,p)$. Verify that $n - X$ has a binomial distribution $B(n, 1-p)$.

3. We randomly draw a point from a disk of radius R. Let X denote the distance of this point from the center of the disk. Find the distribution of X^2.

4. Let X be a random variable with density $g(x) = \frac{1}{2}x\, 1_{[0,2]}(x)$. Find the distribution of $Y = \min\{X-1, 0\}$. Does Y have a density function?

5. Let X be a random variable with values from the set $\{1, 2, \ldots, 10\}$, such that

$$\mathbb{P}(X=1) = \frac{1}{2}, \qquad \mathbb{P}(X=2) = \mathbb{P}(X=3) = \ldots = \mathbb{P}(X=10) = p.$$

Find p, $\mathbb{E}X$ and $\mathbb{E}(4X+5)$.

6. Let X be a random variable from a distribution concentrated over the set $\{1, 2, \ldots, 10\}$, such that

$$\mathbb{P}(X=1) = \frac{1}{2}, \qquad \mathbb{P}(X=2) = \mathbb{P}(X=3) = \ldots = \mathbb{P}(X=10) = p.$$

Calculate p, $\mathbb{E}X$ and $\mathbb{E}(4X+5)$.

7. Let X be a random variable such that $P(X=k) = \frac{1}{n}$ for $k = 2, 4, 6 \ldots, 2n$. Calculate $\mathbb{E}X$ and $\mathbb{E}(2X+1)$.

8. Let X be a random variable from a Binomial distribution with parameters 5 and $\frac{1}{3}$. Find $\mathbb{E}X$ and $\mathbb{E}(4X-1)$.