

Probability Calculus

Anna Janicka

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INDEPENDENCE OF EVENTS

BERNOULLI PROCESS

POISSON THEOREM

Plan for today

1. Independence of events
2. The Bernoulli Process
3. Approximation of the Bernoulli Process for large n – Poisson Theorem



Independence of Events

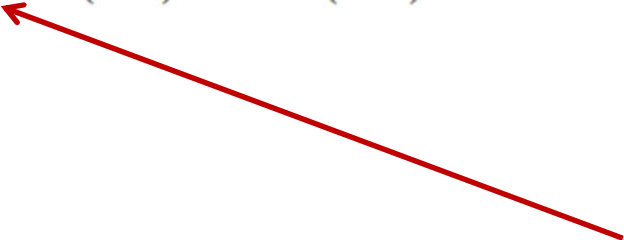
1. Definition

Events A and B are independent,
if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot P(B)$.

2. Examples

- die roll
- choosing a card

Symmetric.
Stochastic
independence



Independence of Events – cont.

3. Independence of 3+ events

Events A_1, A_2, \dots, A_n are independent, if for all indices $1 \leq i_1 < i_2 < \dots < i_k \leq n$, $k = 2, 3, \dots, n$, we have

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdot \dots \cdot \mathbb{P}(A_{i_k}).$$

4. Examples.

- The definition may not be simplified!
- Independence and pairwise independence



Independence of Events – cont. (2)

5. Theorem. Independence conditions

*Let A_1, A_2, \dots, A_n be a sequence of events,
and denote $A_i^0 = A_i, A_i^1 = A_i'$.*

The following conditions are equivalent:

- (i) *events A_1, A_2, \dots, A_n are independent,*
- (ii) *for any sequence $\varepsilon_1, \dots, \varepsilon_n$, where $\varepsilon_i \in \{0, 1\}$ ($i = 1, \dots, n$),
events $B_1 = A_1^{\varepsilon_1}, \dots, B_n = A_n^{\varepsilon_n}$ are independent,*
- (iii) *for any sequence $\varepsilon_1, \dots, \varepsilon_n$, where $\varepsilon_i \in \{0, 1\}$ ($i = 1, \dots, n$),
we have $\mathbb{P}(A_1^{\varepsilon_1} \cap \dots \cap A_n^{\varepsilon_n}) = \mathbb{P}(A_1^{\varepsilon_1}) \cdot \dots \cdot \mathbb{P}(A_n^{\varepsilon_n})$.*



Bernoulli Process

1. Definition

A Bernoulli process is a sequence of n independent repetitions of a single experiment (referred to as a Bernoulli trial) with two possible outcomes: one of these outcomes is referred to as a success (usually denoted as 1), and occurs with probability $p \in [0, 1]$, and the other one is a failure (usually denoted as 0), and occurs with probability $q = 1 - p$.

- a finite or an infinite process



Bernoulli Process – cont.

2. Examples

3. Probability in a Bernoulli process:

$$\Omega = \{(a_1, a_2, \dots, a_n) : a_i \in \{0, 1\}, i = 1, 2, \dots, n\}$$

$$\mathcal{F} = 2^\Omega$$

$$\mathbb{P}(\{(a_1, a_2, \dots, a_n)\}) = p^{\sum_{i=1}^n a_i} (1 - p)^{n - \sum_{i=1}^n a_i}$$

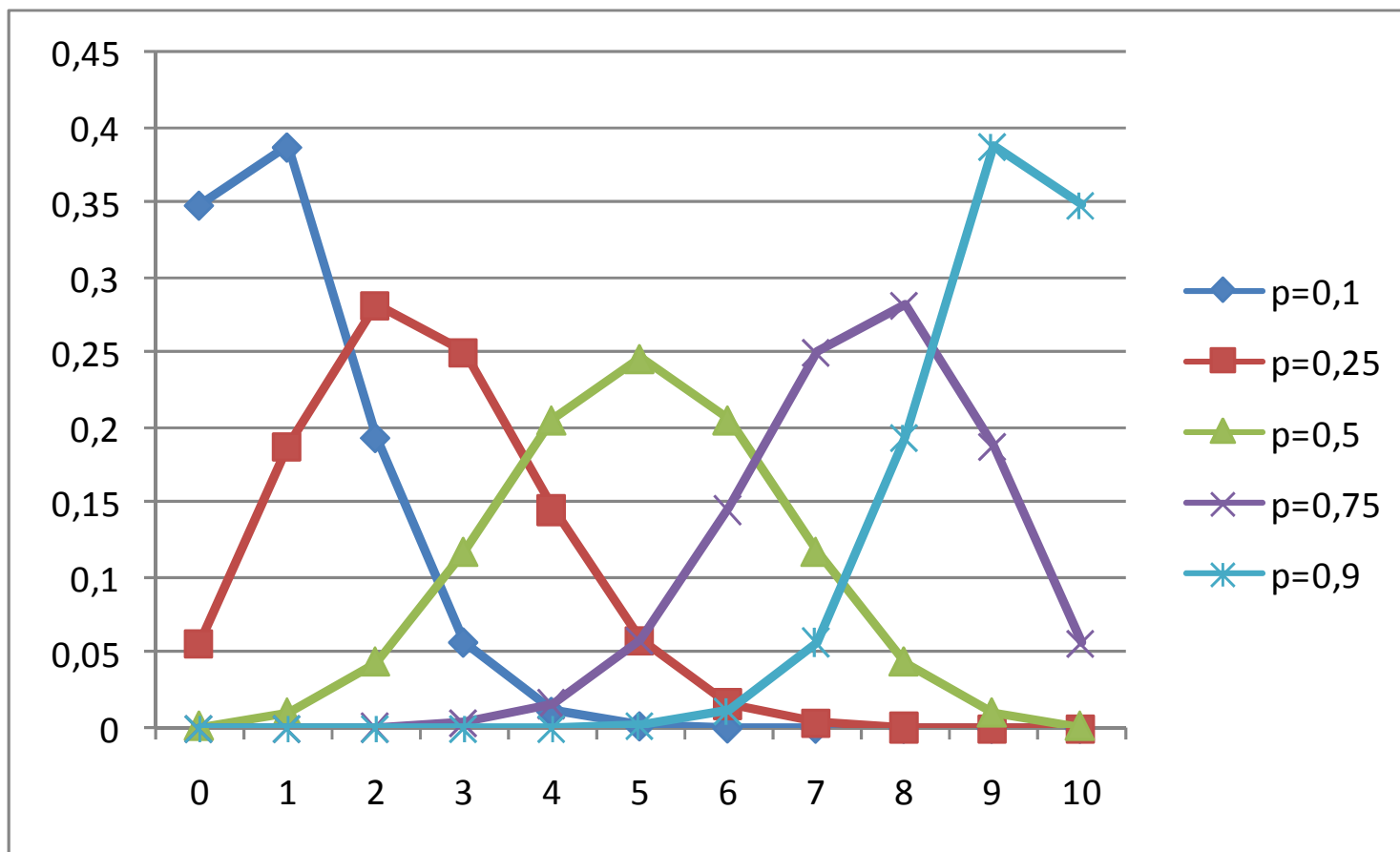
- probability of exactly k successes in n trials

$$\binom{n}{k} p^k (1 - p)^{n-k}$$



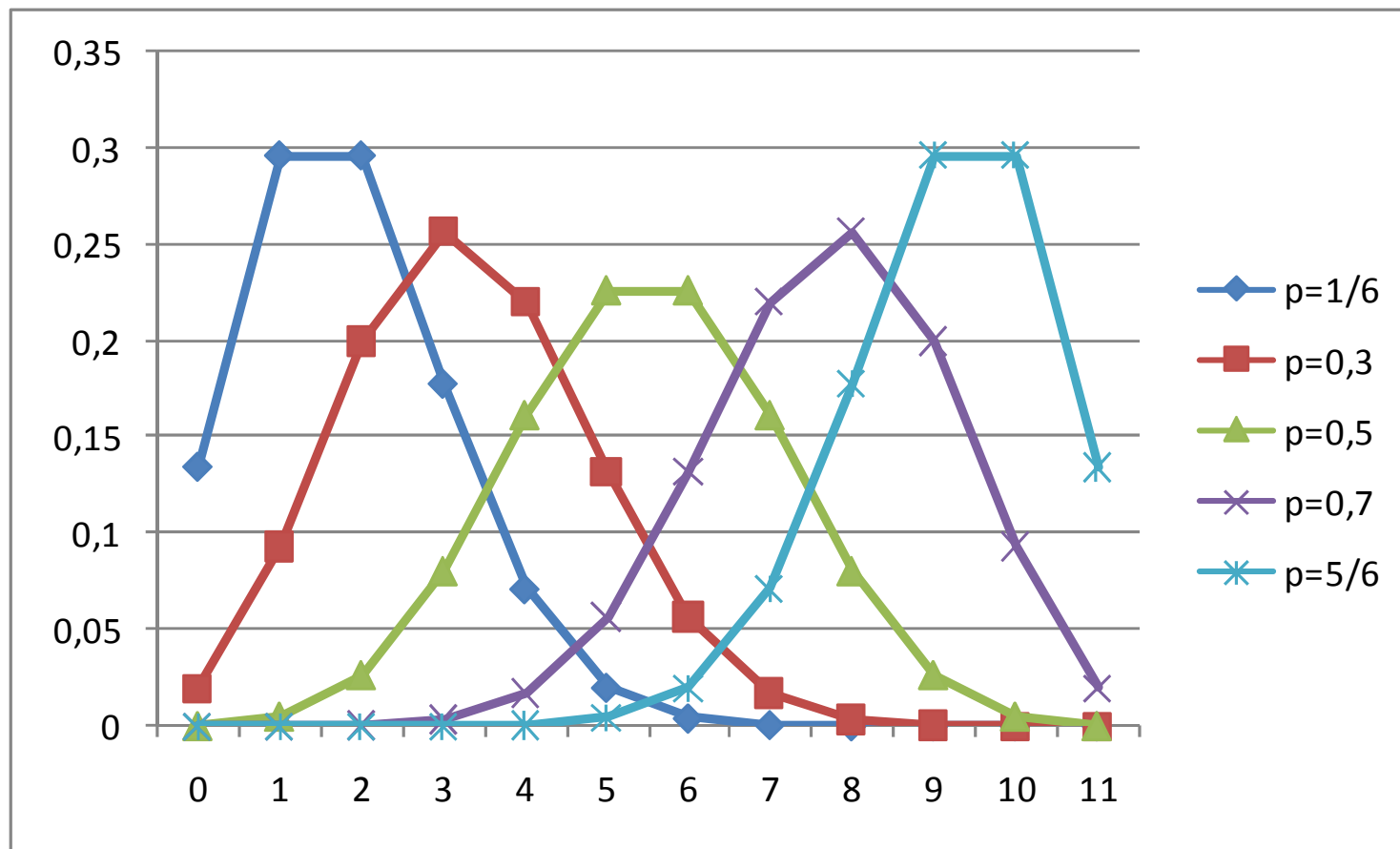
Bernoulli Process – cont. (2)

$n=10$



Bernoulli Process – cont. (3)

$n=11$



Bernoulli Process – cont. (4)

4. Examples

- coin flip
- die roll

5. The most probable number of successes

6. Infinite sequence of heads



Poisson Theorem

1. Poisson Theorem

If $p_n \in [0, 1]$, $\lim_{n \rightarrow \infty} np_n = \lambda > 0$,

then for $k = 0, 1, 2, \dots$,

we have that

$$\lim_{n \rightarrow \infty} \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$$

2. Assessment of approximation error

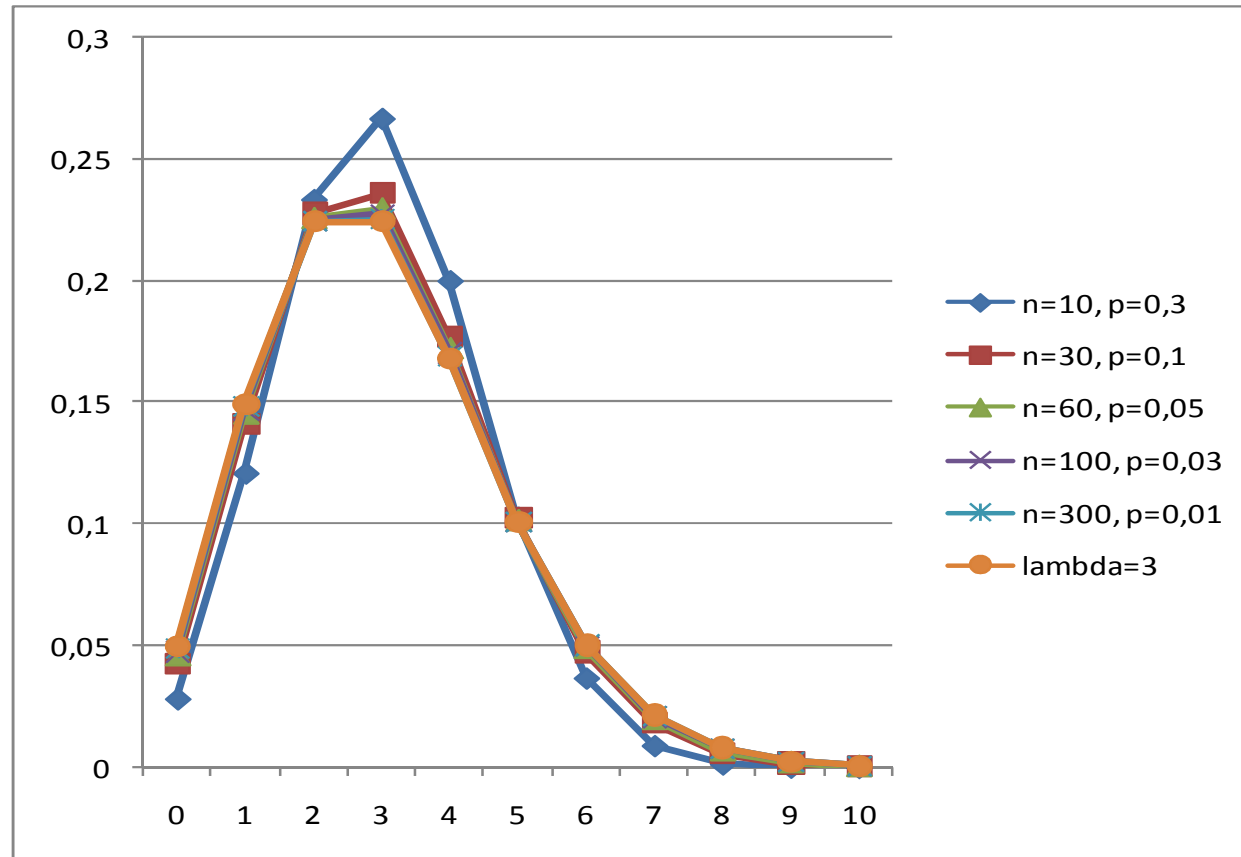
Let S_n denote the number of successes in a Bernoulli process with n trials and a probability of success in a single trial equal to p , and let $\lambda = np$. For any $A \subset \{0, 1, 2, \dots\}$, we have

$$\left| \mathbb{P}(S_n \in A) - \sum_{k \in A} \frac{\lambda^k}{k!} e^{-\lambda} \right| \leq \frac{\lambda^2}{n}.$$



Poisson Theorem – cont.

The Poisson and Bernoulli processes



Poisson Theorem – cont. (2)

3. Examples of good and not so good approximations





WARSAW UNIVERSITY
Faculty of Economic Sciences