

# Probability Calculus

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**INTRODUCTION TO PROBABILITY – CONT.**

**CONDITIONAL PROBABILITY**

# Plan for today

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1. Sample spaces and basic properties of probability – cont.
2. Conditional probability



# Examples

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1. Countable sample spaces
2. Geometric probability



# Basic properties of probability

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## □ Theorem 1 (arithmetics)

Theorem 1. Let  $A, B, A_1, A_2, \dots \in \mathcal{F}$ . Then

- (i)  $\mathbb{P}(\emptyset) = 0$ ,
- (ii) If  $A_1, A_2, \dots, A_n$  are pairwise disjoint, then  $\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i)$ .
- (iii)  $\mathbb{P}(A') = 1 - \mathbb{P}(A)$ .
- (iv) If  $A \subseteq B$ , then  $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$ .
- (v) If  $A \subseteq B$ , then  $\mathbb{P}(A) \leq \mathbb{P}(B)$ .
- (vi)  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ .
- (vii)  $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ .



## Examples – cont.

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1. Symmetric coin toss, asymmetric coin toss
2. Dice rolling
- 3. Classic scheme (simple probability)**
4. Drawing cards
5. Countable sample spaces
6. Geometric probability



## Basic properties of probability – cont.

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### □ Theorem 2 (inclusion-exclusion principle)

If  $A_1, A_2, \dots, A_n \in \mathcal{F}$ , then

$$\begin{aligned} \mathbb{P}(A_1 \cup A_2 \cup \dots \cup A_n) &= \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i < j} \mathbb{P}(A_i \cap A_j) + \dots \\ &\quad + (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) \end{aligned}$$



## Examples – cont.

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1. Symmetric coin toss, asymmetric coin toss
2. Dice rolling
3. **Classic scheme (simple probability)**
4. Drawing cards
5. Countable sample spaces
6. Uncountable spaces. Geometric probability



## Further properties of probability

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### □ Definitions of contracting and expanding sets

*Assume  $A_1, A_2, \dots$  is a sequence of events.*

*We will call this sequence **expanding** if*

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots,$$

*and **contracting** if*

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$$



## Further properties of probability – cont.

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### □ Theorem: Rule of Continuity

*Assume that  $(A_n)_{n=1}^{\infty}$  is a sequence of events.*

*(i) If the series is expanding, then*

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \mathbb{P} \left( \bigcup_{n=1}^{\infty} A_n \right).$$

*(ii) If the series is contracting, then*

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \mathbb{P} \left( \bigcap_{n=1}^{\infty} A_n \right).$$



# Conditional probability

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## 1. Intuition

- New product marketing
- Results of dice rolls when only the sum is known

## 2. Definition

*Let  $X$  and  $Y$  be events, such that  $\mathbb{P}(Y) > 0$ .  
By a **conditional probability** of event  $X$   
under the condition  $Y$  we will understand*

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(X \cap Y)}{\mathbb{P}(Y)}.$$



## Conditional probability – cont.

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3. Conditional probability is probability

4. Theorem (Chain rule)

*For any sequence of events  $A_1, \dots, A_n$ , such that  $\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_{n-1}) > 0$ , we have*

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) =$$

$$\mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$



## Conditional probability – cont. (2)

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5. Example (Drawing successive aces)
6. Definition of partition

*Any family of events  $\{H_i\}_{i \in I}$ ,  
such that  $H_i \cap H_j = \emptyset$  for  $i \neq j$   
and  $\bigcup_{i \in I} H_i = \Omega$  is called a  
**partition** of the sample space  $\Omega$ .*

A finite, countable partition

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## Conditional probability – cont. (3)

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### 7. Theorem (Law of Total Probability)

*For any finite partition  $\{H_1, H_2, \dots, H_n\}$  of the sample space  $\Omega$ , such that all  $H_i$  have positive probability, and for any event  $A$ , we have*

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A|H_i) \cdot \mathbb{P}(H_i).$$

### 8. Examples

- Car manufacturer
- Balls in a box



## Conditionat probability – cont. (4)

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### 9. Theorem (Bayes' Rule)

*Let  $\{H_i\}_{i \in I}$  be a countable (finite or infinite) partition of  $\Omega$  into sets of positive probability. For any event  $A$  of positive probability, we have*

$$\mathbb{P}(H_j|A) = \frac{\mathbb{P}(A|H_j)\mathbb{P}(H_j)}{\sum_{i \in I} \mathbb{P}(A|H_i)\mathbb{P}(H_i)}.$$

### 10. Examples





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