# Probability Calculus

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## INTRODUCTION TO PROBABILITY CALCULUS

## Some technicalities

- Contact: ajanicka@wne.uw.edu.pl
- Office hours: Tue, 9:15 AM, FoES room B006
- web page (materials):

  www.wne.uw.edu.pl/azylicz
- Readings:
  - **Charles M. Grinstead and J. Laurie Snell, Introduction to Probability, available online**
  - **Sheldon M. Ross, Introduction to Probability Models, available in the FoES library and online**
  - **Wackerly, D., Mendenhall, W., & Scheaffer, R. Mathematical statistics with applications, available in the FoES library**

# Assessment

1. Presence during lectures – recommended; presence during classes – mandatory
2. Class assessment: Necessary condition: at least 6 out of 7 short tests passed + activity during classes
3. Homework
4. Lecture assessment: Two tests, one on December 7th, 6:30 PM (40pts), one at the end of the semester (50pts) + homework (10pts)
5. Evaluation 1st period: Test1 + Test2 + homework
6. Evaluation 2nd period: max {Test1 + Test2b + homework, 9/5 Test2b + homework}

## What to expect

- Lecture notes (web page)
- Problems to solve during classes (web page)
- Homework (web page)
- Preparation for classes: previous lecture material + previous classes

# Probability Calculus 1st term exam results and homework

| Grade | Average no of homework points |
|---|---|
| 2 | 7.598795 |
| 3 | 8.677941 |
| 3.5 | 8.994286 |
| 4 | 9.394737 |
| 4.5 | 9.56 |
| 5 | 9.866667 |
| Total | 8.456561 |

….
Is what we see a coincidence?
Is there a relationship between the grade and the number of homework points?

Examples of other questions:
What is the probability that a student will pass the exam if the number of homework points is equal to 10? 0?
What is the chance that everybody will pass?
What does the distribution of grades depend on?

## Thematic scope of course

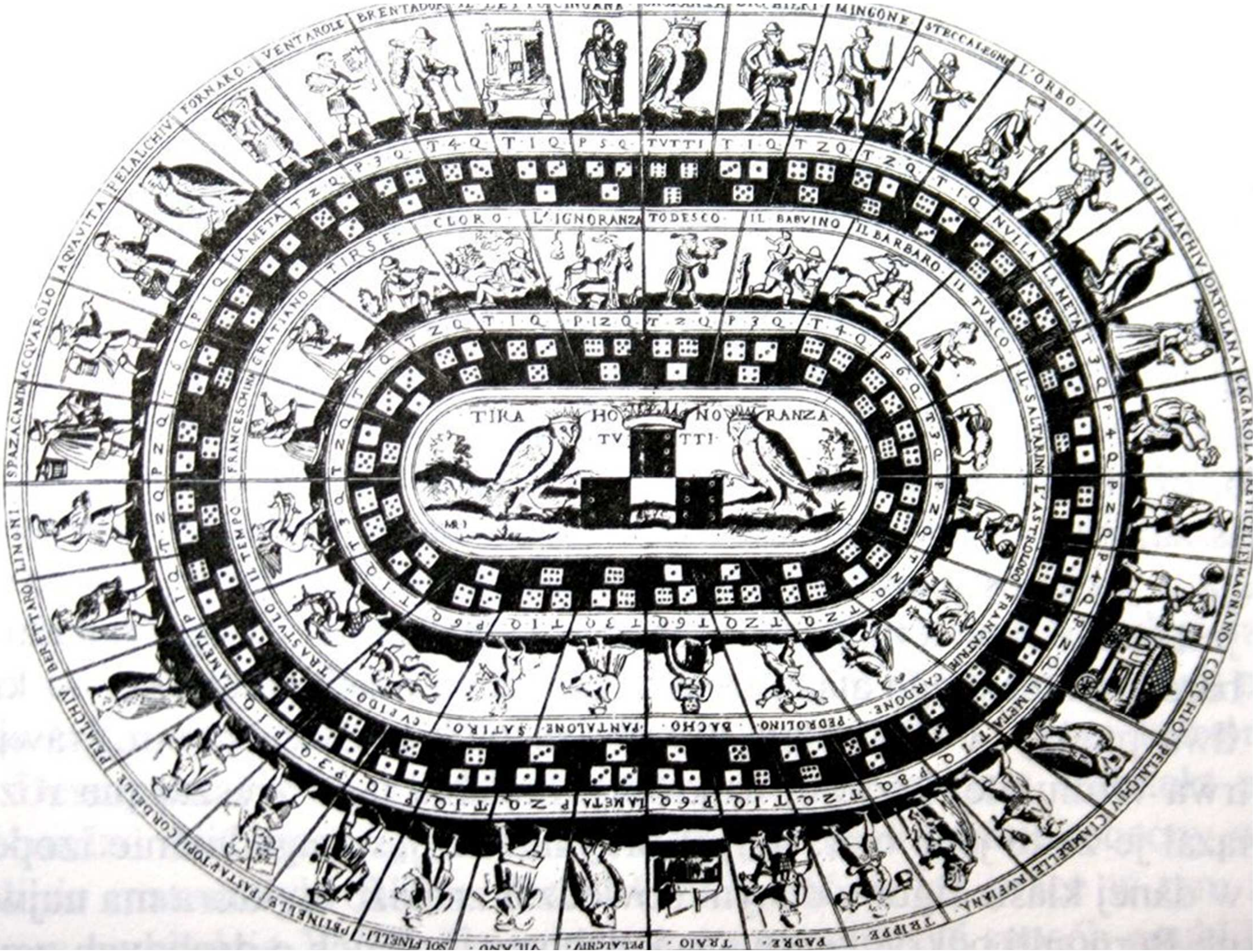
- Some basics and „classics”
- „Contemporary” probability
- Reality description – random variables. Crucial in statistics and econometrics
- Limit theorems – crucial as above, very important in practice (e.g. insurance)

## Plan for today

1. Historical perspective
2. Basic schemes
3. Basic definitions and notations, examples
4. *$\sigma$-algebras*
5. Probability intuitively and **Kolmogorov axioms,** Examples
6. Basic properties of probability

## 1. Historical perspective

- Motivation:
  - gambling
  - statistics of births and deaths
  - insurance of transports
- „Paradoxes”
- First mathematical publications without errors: Bernoulli, 1752
- „Contemporary probability”: Kolmogorov axioms, 1933



TIRA
HO
NO
RANZA
TVTTI
VENTAROLE
BRENTADOR
CINGANA
MINGONE
STECCALEGNE
L'ORBO
IL MATTO
PELACHIV
ORTOLANA
CAGAROLA
MAGNANO
COCCHIO
CIAMBELLE
RVFFIANA
TRIPPE
PADRE
VILANO
SOLFINELLI
PETINELLE
CORDELLE
BERETTARO
LIMONI
SPAZACAMIN
ACQVAROLO
AQVAVITA
PELALCHIV
FORNARO
CLORO
L'IGNORANZA
TODESCO
IL BABVINO
IL BARBARO
IL TVRCO
IL SALTARINO
L'ASTROLOGO
CARDONE
PEDROLINO
BACHO
PANTALONE
SATIRO
CVPIDO
TRASTVLLO
IL TEMPO
GRATIANO
TIRSE
LA META
NVLLA

## 1. Historical perspective

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## Gerolamo Cardano, De Ludo Aleae (Book on Games of Chance), 1564

*"If it is necessary for someone that he should throw at least twice, then you know that the throws favorable for it are 91 in number, and the remainder is 125; so we multiplying each of these numbers by itself and get to 8,281 and 15,625, and the odds are about 2 to 1."*

*"**This reasoning seems to be false**... for example, the chance of getting one of any three chosen faces in one cast of one dice is equal to the chance of getting one of the other three, but according to this reasoning there would be an even chance of getting a chosen face each time in two casts, and thus in three, and four, **which is most absurd**."*

## 1. Historical perspective

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## 2. Basic schemes

1. Variations with repetitions $n^k$
2. Variations without repetitions $n!/(n-k)!$
3. Permutations $n!$
4. Combinations $\binom{n}{k}$

## 3. Basic definitions and notations

Elementary event: $\omega$

Sample space : $\Omega$

Event: *A, B, etc.*

Special events, operations:

$\Omega$, , $A'$, $A \cup B$, $A \cap B$, $A \setminus B$, $A \subseteq B$

## 3. Examples

1. Coin toss
2. Dice rolling
3. Rolling of a pair of dice – sum of points
4. Draw of 13 cards out of 52 – with and without order
5. Coin toss until first „heads"
6. Needle on a table

## 4. $\sigma$-*algebra*

Defines the sets that we can measure (calculate probability). In most simple cases: **we don't need to worry about it.**

*Definition of a $\sigma$-algebra F of subsets of $\Omega$*

*A family $\mathcal{F}$ of subsets of $\Omega$ is called a* $\sigma$-algebra, *if*

$$(i) \quad \emptyset \in \mathcal{F},$$

$$(ii) \quad A \in \mathcal{F} \Rightarrow A' \in \mathcal{F},$$

$$(iii) \quad A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

# 5. Probability intuitively – frequencies

$$\rho_n(A) = \frac{\text{number of occurrences of } A}{n}$$

- Calculating frequencies
- Properties of frequencies
- Limit =?

## 5. Probability formally - Kolmogorov Axioms

- For a given ($\Omega$, F) we define probability as a function satisfying 3 conditions

$$
\begin{aligned}
(i) \quad & 0 \leq \mathbb{P}(A) \leq 1, \\
(ii) \quad & \mathbb{P}(\Omega) = 1, \\
(iii) \quad & \text{if } A_1, A_2, \ldots \in \mathcal{F} \text{ are pairwise disjoint} \\
& \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).
\end{aligned}
$$

- Probability space ($\Omega$, F, P )

## 5. Examples

1. Symmetric coin toss, asymmetric coin toss
2. Dice rolling
3. **Classic scheme (simple probability)**
4. Drawing cards
5. Countable sample spaces
6. Geometric probability

# 5. Basic properties of probability

## □ Theorem 1 (arithmetics)

**Theorem 1.** *Let* $A, B, A_1, A_2, \ldots \in \mathcal{F}$. *Then*

(i) $\mathbb{P}(\emptyset) = 0,$

(ii) *If* $A_1, A_2, \ldots, A_n$ *are pairwise disjoint, then* $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$

(iii) $\mathbb{P}(A') = 1 - \mathbb{P}(A).$

(iv) *If* $A \subseteq B$, *then* $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A).$

(v) *If* $A \subseteq B$, *then* $\mathbb{P}(A) \leqslant \mathbb{P}(B).$

(vi) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$

(vii) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} \mathbb{P}(A_i).$

## 5. Examples – cont.

1. Symmetric coin toss, asymmetric coin toss
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3. **Classic scheme (simple probability)**
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# 5. Basic properties of probability – cont.

- Theorem 2 (inclusion-exclusion principle)

*If* $A_1, A_2, \ldots, A_n \in \mathcal{F}$, *then*

$$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \ldots + (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n)$$

## 5. Examples – cont.

1. Symmetric coin toss, asymmetric coin toss
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