Mathematical Statistics 2018/2019, Problem set 8

Estimator properties: revision

1. Let $X_1, \ldots, X_n$ be a sample of independent observations from an exponential distribution with an unknown parameter $\lambda > 0$.

   - Find $\hat{\lambda}_{ML}$, the maximum likelihood estimator for λ.
   - Verify that $\hat{\lambda}_{ML}$ is biased, and propose $\hat{\lambda}_U$, an unbiased estimator on the base of $\hat{\lambda}_{ML}$.

     Hint. If $X_1, \ldots, X_n$ are independent random variables from an exponential distribution with parameter λ, $Z = \sum_{i=1}^n X_i$ has a distribution with density $f_Z(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}$ for $x > 0$.

     Hint 2. $\int_0^\infty x^k e^{-\lambda x} dx = \frac{k!}{\lambda^{k+1}}$ for integer values of k.
   - Compare $\hat{\lambda}_{ML}$ and $\hat{\lambda}_U$ on the base of the MSE.
   - Calculate the efficiency of $\hat{\lambda}_U$. Is this estimator efficient?
   - Verify whether $\hat{\lambda}_{ML}$ and $\hat{\lambda}_U$ are consistent.
   - Is $\hat{\lambda}_{ML}$ asymptotically normal? If yes, is it asymptotically efficient?

2. Let $X_1, \ldots, X_n$ be a sample of independent observations from an exponential distribution with parameter $\frac{1}{\lambda}$, where $\lambda > 0$ is unknown. Find a such that the estimator $\hat{\lambda}_a = a \sum_{i=1}^n X_i$ has the smallest MSE. Is this estimator biased? Is this estimator consistent?

3. Let $X_1, \ldots, X_n$ be a sample of independent observations from a distribution with density equal to $f_\theta(x) = \frac{\theta}{x^{\theta+1}}$ for $x > 1$ and 0 otherwise, where $\theta > 2$ is an unknown parameter. Find the ML estimator of θ. Determine whether this estimator is: consistent? asymptotically normal? If yes, find the normal distribution that best resembles the distribution of the estimator for large n. *Hint. The expected value for a random variable with density f_θ is equal to $\frac{\theta}{\theta-1}$.*

4. Let $X_1, \ldots, X_n$ be a sample of independent observations from a geometric distribution with unknown parameter $\theta \in (0, 1)$, i.e. such that $P(X = x) = \theta(1-\theta)^x$ for $x = 0, 1, \ldots$. Find the maximum likelihood estimator for θ and the method of moments estimator for θ (based on the mean) and compare the two estimators.

   Hint. In a geometric distribution, $EX = \frac{1-\theta}{\theta}$.