Mathematical Statistics

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Bayesian Statistics

Plan for Today

1. Bayesian Statistics
   - a priori and a posteriori distributions
   - Bayesian estimation:
     - Maximum a posteriori probability (MAP)
     - Bayes Estimator
   - Credible intervals

Bayesian Statistics vs. traditional statistics

Frequentist: unknown parameters are given (fixed), observed data are random

Bayesian: observed data are given (fixed), **parameters are random**

Bayesian Statistics

Our knowledge about the unknown parameters is described by means of probability distributions, and additional knowledge may affect our description.

Knowledge:

- general
- specific

Example: coin toss

Bayesian Model

- $X_1, \ldots, X_n$ come from distribution P_θ, with density $f_\theta(\mathbf{x})$ – conditional density given a specific value of θ (likelihood function).
- P – family of probability distributions P_θ, indexed by the parameter $\theta \in \Theta$
- General knowledge: distribution Π over the parameter space Θ, given by $\pi(\theta)$ – the so-called **a priori/prior** distribution of θ,

$$\theta \sim \Pi$$

Bayesian Model – cont.

Additional knowledge (specific, contextual): based on observation. We have a joint distribution of observations and θ:

$$f(x_1, x_2, \dots, x_n, \theta) = f(x_1, x_2, \dots, x_n \mid \theta)\pi(\theta)$$

on this basis we can derive the conditional distribution of θ (given the observed data)

$$\pi(\theta \mid x_1, \dots, x_n) = \frac{f(x_1, \dots, x_n \mid \theta)\pi(\theta)}{m(x_1, \dots, x_n)},$$

where $$m(x_1, \dots, x_n) = \int_\Theta f(x_1, \dots, x_n \mid \theta)\pi(\theta)d\theta$$

is a marginal distribution for the obs.

Bayesian Model – a posteriori distribution

$\pi(\theta \mid x_1,\ldots,x_n)$ is called the **a posteriori/ posterior** distribution, denoted Π_x

The posterior distribution reflects all knowledge: general (initial) and specific (based on the observed data).

Grounds for Bayesian inference and modeling

A priori and a posteriori distributions: examples

1. Let $X_1, \ldots, X_n$ be IID r.v. from a 0-1 distr. with prob. of success θ, let

$$\pi(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}$$

for $\theta \in (0,1)$

Beta(α,β) distr with mean = $\alpha/(\alpha+\beta)$

where $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

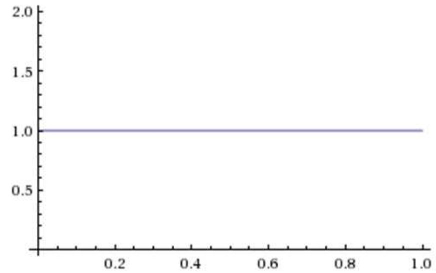
and $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}\exp(-u)du = (\alpha-1)\Gamma(\alpha-1)$

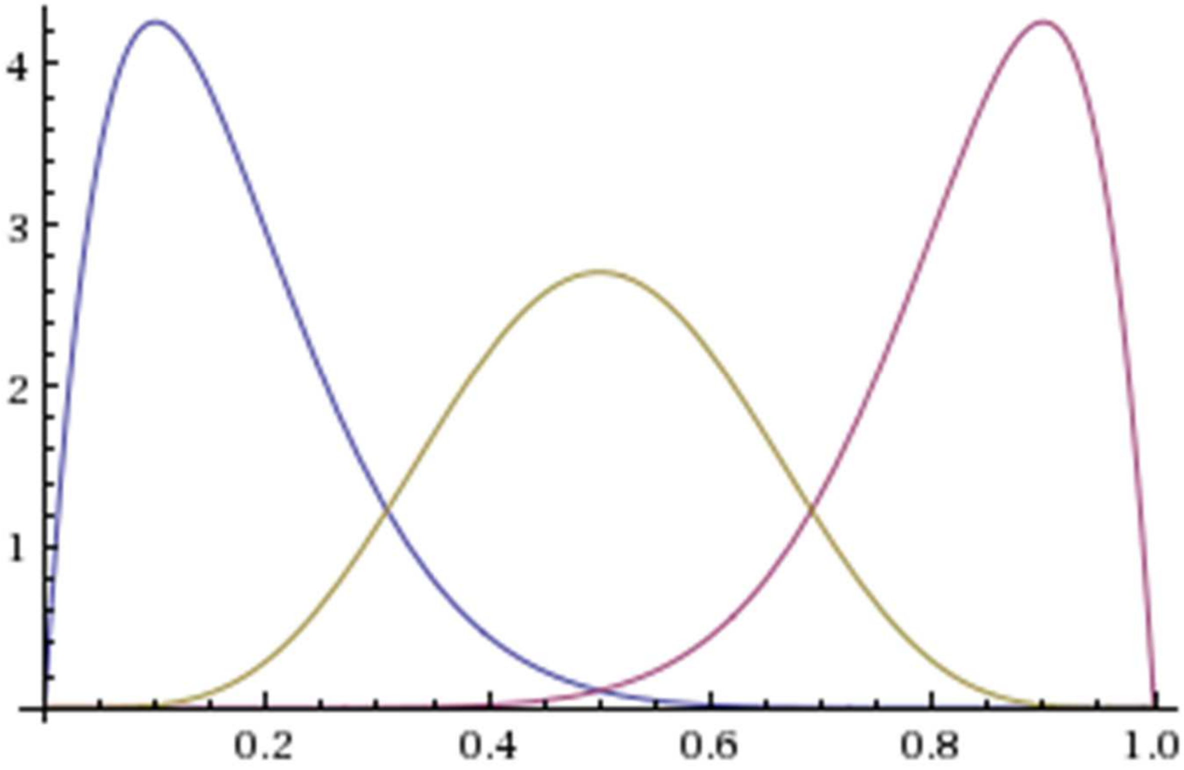
then the posterior distribution:

$$\text{Beta}\left(\sum_{i=1}^n x_i + \alpha, n - \sum_{i=1}^n x_i + \beta\right)$$

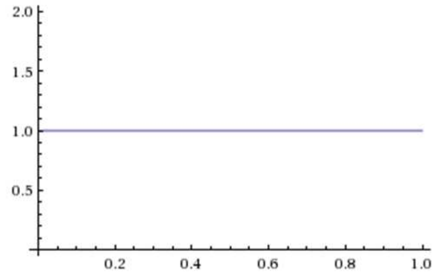
conjugate prior for Bernoulli distr.

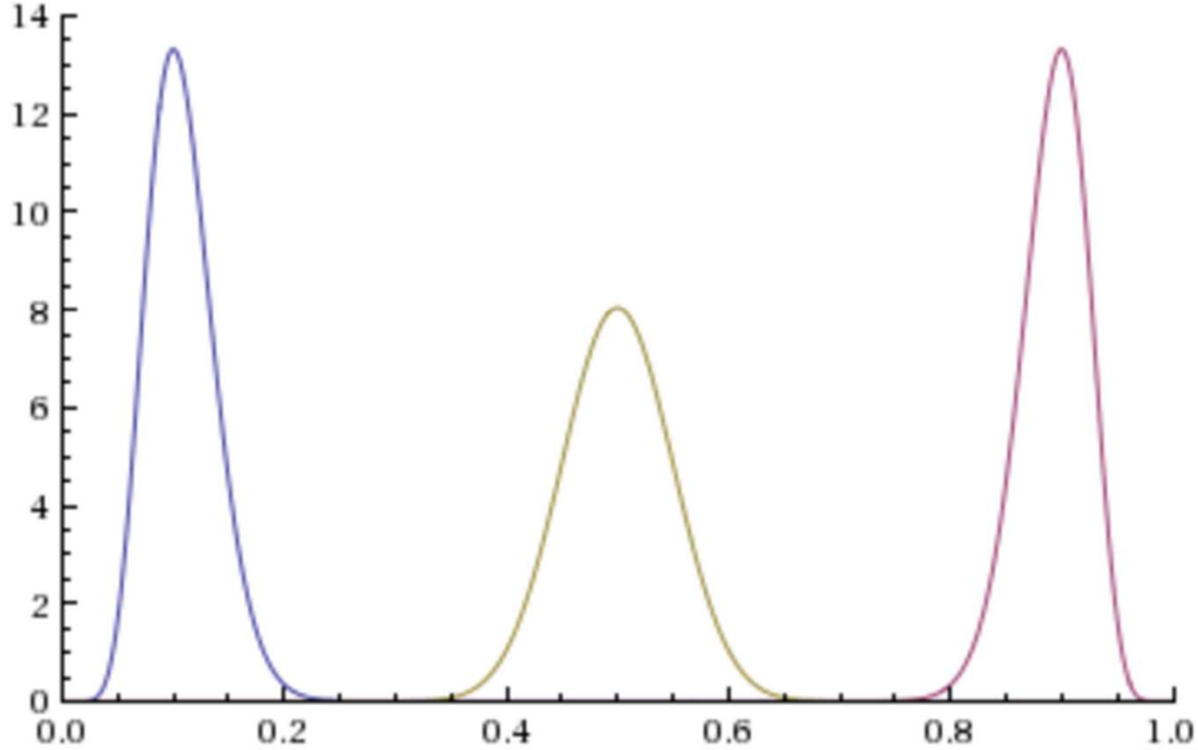
For a Beta (1,1) prior and data: n=10 and 1, 5, 9 successes



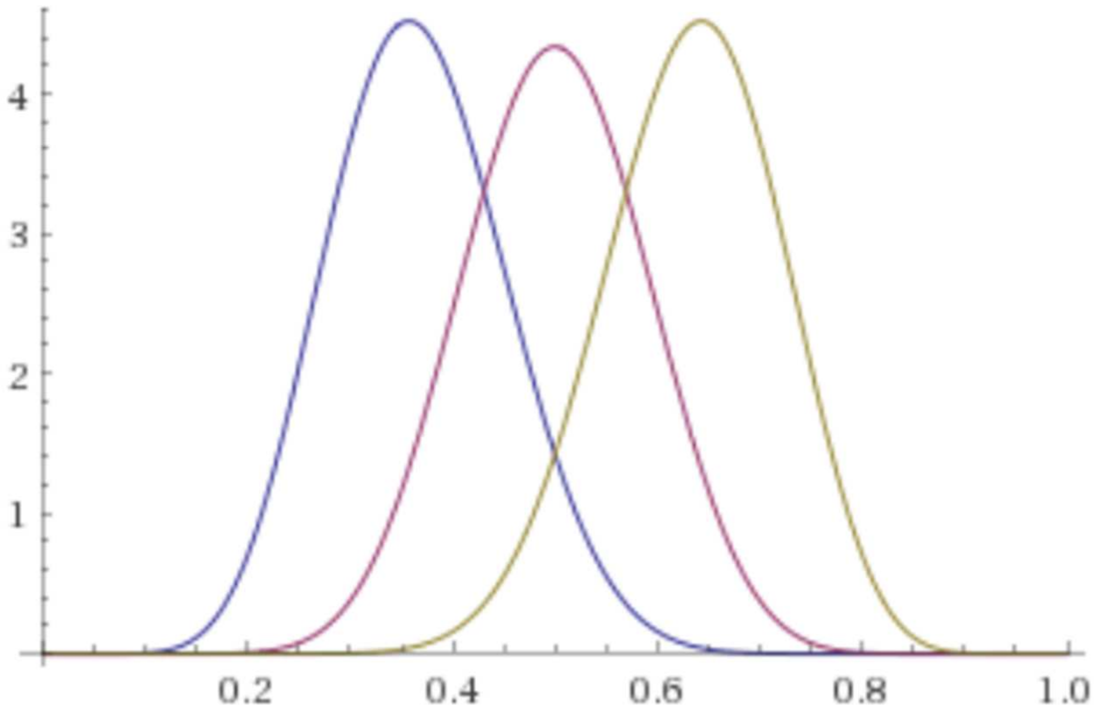


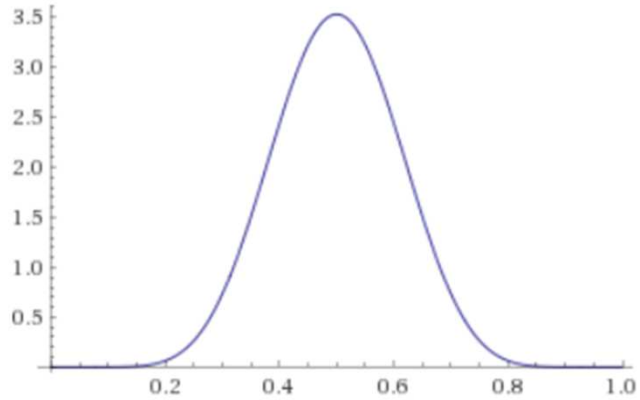
For a Beta (1,1) prior and data: n=100 and 10, 50, 90 successes



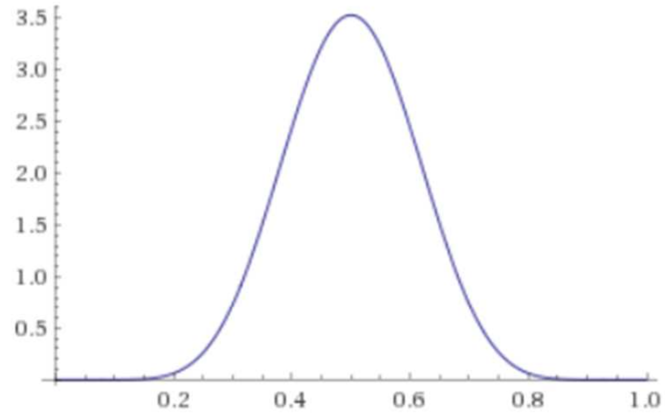


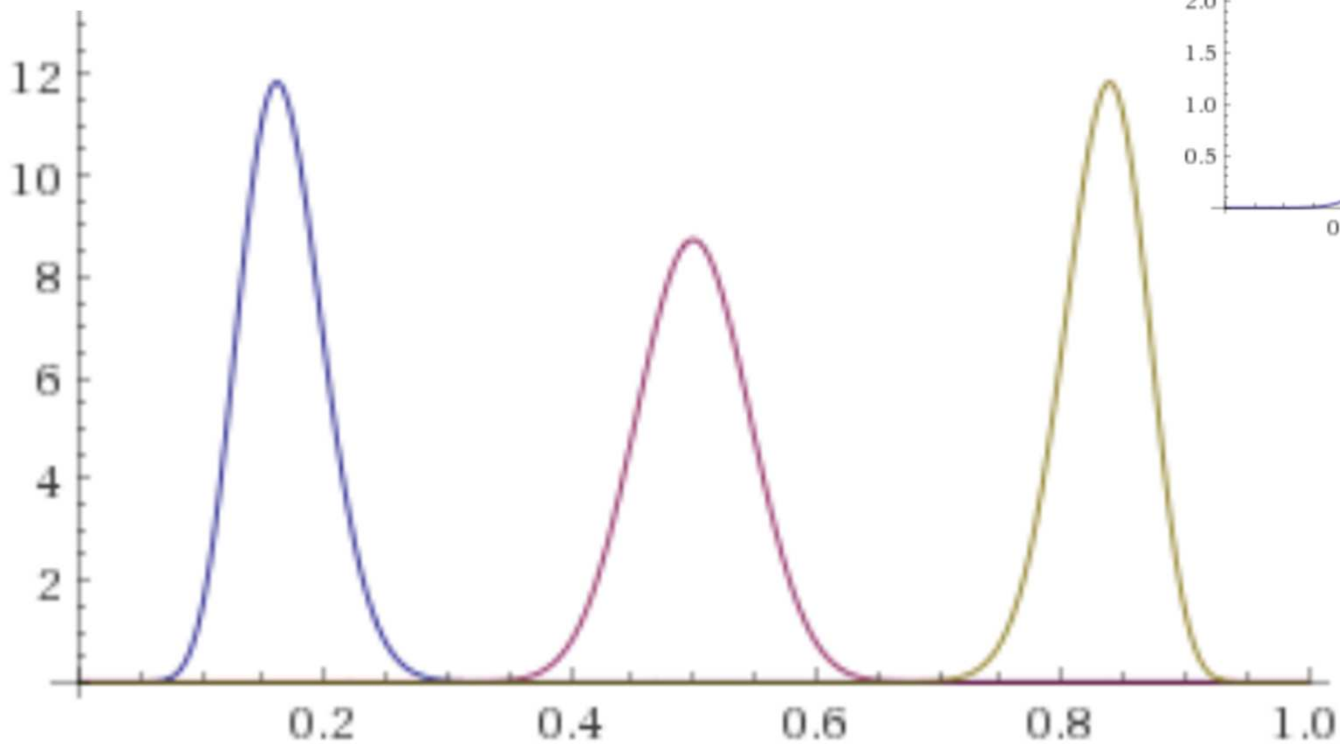
For a Beta (10,10) prior and data: n=10 and 1, 5, 9 successes



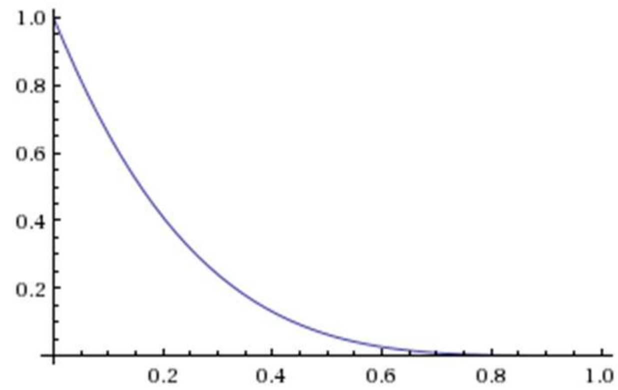


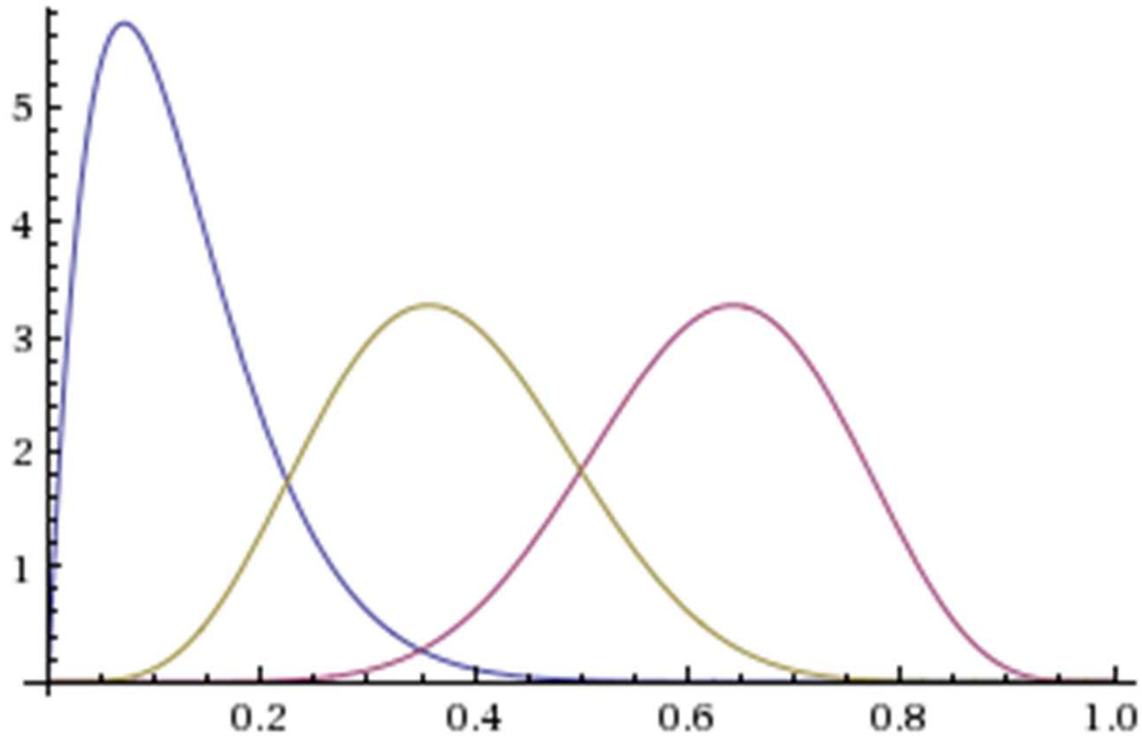
For a Beta (10,10) prior and data: n=100 and 10, 50, 90 successes



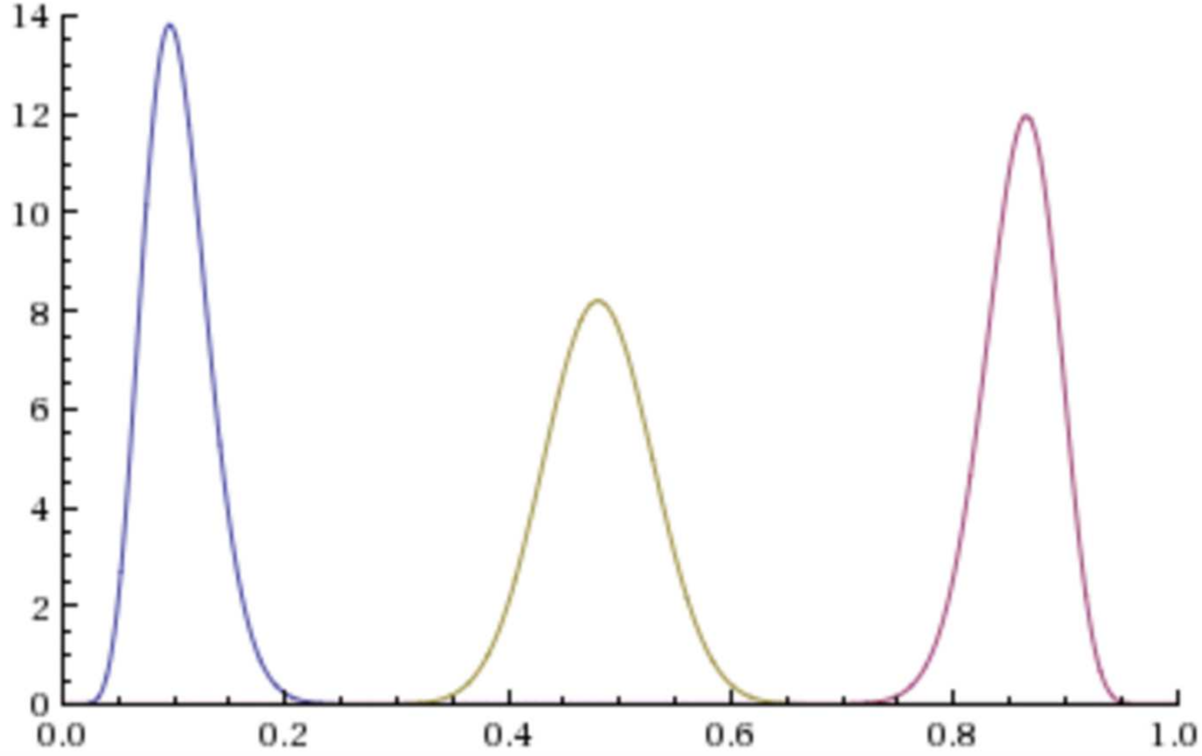


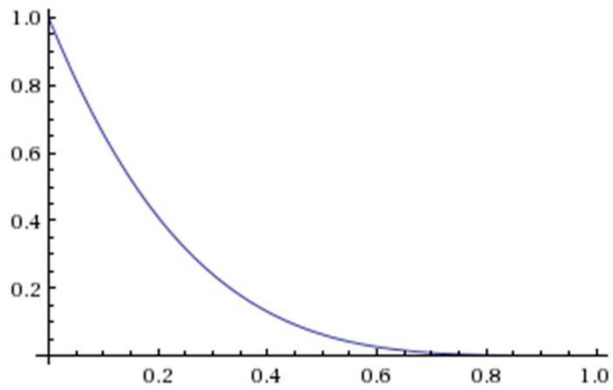
For a Beta (1,5) prior and data: n=10 and 1, 5, 9 successes





For a Beta (1,5) prior and data: n=100 and 10, 50, 90 successes





A priori and a posteriori distributions: examples (2)

2. Let $X_1, \ldots, X_n$ be IID r.v. from $N(\theta, \sigma^2)$, and σ^2 known; $\theta \sim N(m, \tau^2)$ for m, τ known.

Then the posterior distribution for θ:

$$N\left(\frac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}, \frac{1}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right)$$

conjugate prior for a normal distr.

Bayesian Statistics

Based on the Bayes approach, we can

- find estimates
- find an equivalent of confidence intervals
- verify hypotheses

- make predictions

Bayesian Most Probabale (BMP) / Maximum a posteriori Probability (MAP) estimate

Similar to ML estimation: the argument which maximizes the posterior distribution:

$$\pi(\hat{\theta}_{BMP} \mid x_1,\dots,x_n) = \max_{\theta} \pi(\theta \mid x_1,\dots,x_n)$$

i.e.

$$BMP(\theta) = \hat{\theta}_{BMP} = \operatorname{argmax}_{\theta} \pi(\theta \mid x_1,\dots,x_n)$$

BMP: examples

1. Let $X_1, \dots, X_n$ be IID r.v. from a Bernoulli distr. with prob. of success θ; for $\theta \in (0,1)$

$$\pi(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}$$

We know the posterior distribution:

$$\text{Beta}\left(\sum_{i=1}^{n} x_i + \alpha, n - \sum_{i=1}^{n} x_i + \beta\right)$$

Beta(α,β) distr; the mode of this distr = $(\alpha-1)/(\alpha+\beta-2)$ for $\alpha>1$, $\beta>1$

we have max for

$$BMP(\theta) = \frac{\sum_{i=1}^{n} x_i + \alpha - 1}{n + \beta + \alpha - 2}$$

i.e. for 5 successes in 10 trials for an a priori U(0,1) (i.e. Beta(1,1) distr.), we have $BMP(\theta)=5/10 = ½$

and for 9 successes in 10 trials for the same a priori distr., we have $BMP(\theta)=9/10$

BMP: examples (2)

2. Let $X_1, ..., X_n$ be IID r.v. from $N(\theta, \sigma^2)$, with σ^2 known; $\theta \sim N(m, \tau^2)$ for m, τ known.

Then the posterior distr. for θ: $N\left(\frac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}, \frac{1}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right)$

so

$$BMP(\theta) = \frac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}$$

i.e. if we have sa sample of 5 obs 1.2; 1.7 ; 1.9 ; 2.1; 3.1 from distr. N(θ, 4) and the a priori distr is $\theta \sim N(1, 1)$, then

$$BMP(\theta) = (5/4 * 2 + 1)/(5/4 + 1) = 14/9 \approx 1.56$$

and if the a priori distr were $\theta \sim N(3, 1)$, then

$$BMP(\theta) = (5/4 * 2 + 1*3)/(5/4 + 1) = 22/9 \approx 2.44$$

Bayes Estimator

An estimation rule which minimizes the posterior expected value of a loss function

$L(\theta, a)$ – **loss function,** depends on the true value of θ and the decision a.

e.g. if we want to estimate $g(\theta)$:

$L(\theta, a) = (g(\theta) - a)^2$ – quadratic loss function

$L(\theta, a) = |g(\theta) - a|$ – module loss function

Bayes Estimator – cont.

We can also define the **accuracy of an estimate** for a given loss function :

$$acc(\Pi, \hat{g}(x)) = E\left(L(\theta, \hat{g}(x)) \mid X = x\right) = \int_{\Theta} L(\theta, \hat{g}(x)) \pi(\theta \mid x) d\theta$$

(the average loss of the estimator for a given a priori distribution and data, i.e. for a specific posterior distribution)

Bayes Estimator – cont. (2)

The **Bayes Estimator** for a given loss function $L(\theta, a)$ is $\hat{g}_B$ such that

$$\forall x \quad acc(\Pi, \hat{g}_B(x)) = \min_a acc(\Pi, a)$$

For a quadratic loss function $(\theta - \mathrm{a})^2$:

$$\hat{\theta}_B = E(\theta \mid X = x) = E(\Pi_x)$$

For a module loss function $|\theta - \mathrm{a}|^2$:

$$\hat{\theta}_B = Med(\Pi_x)$$

more generally: $E(g(\theta)|x)$

Bayes Estimator: Example (1)

1. Let $X_1, \dots, X_n$ be IID r.v. from a Bernoulli distr. with prob. of success θ; for $\theta \in (0,1)$

$$\pi(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}$$

We know the posterior distribution:

$$\text{Beta}\left(\sum_{i=1}^{n} x_i + \alpha, n - \sum_{i=1}^{n} x_i + \beta\right)$$

Beta(α,β) distr with mean $= \alpha/(\alpha+\beta)$

so the Bayes Estimator is

$$\hat{\theta}_B = \frac{\sum_{i=1}^{n} x_i + \alpha}{n + \beta + \alpha}$$

i.e. for 5 successes in 10 trials for an a priori U(0,1) (i.e. Beta(1,1) distr.), we have $\hat{\theta}_B = 6/12 = ½$

and for 9 successes in 10 trials for the same a priori distr., we have $\hat{\theta}_B = 10/12 = 5/6$

BMP: examples

1. Let $X_1, ..., X_n$ be IID r.v. from a Bernoulli distr. with prob. of success θ; for $\theta \in (0,1)$

$$\pi(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}$$

We know the poster distribution:

$$\text{Beta}\left(\sum_{i=1}^{n} x_i + \alpha, n - \sum_{i=1}^{n} x_i + \beta\right)$$

Beta(α,β) distr; the mode of this distr = $(\alpha-1)/(\alpha+\beta-2)$ for $\alpha>1$, $\beta>1$

we have max for

$$BMP(\theta) = \frac{\sum_{i=1}^{n} x_i + \alpha - 1}{n + \beta + \alpha - 2}$$

i.e. for 5 successes in 10 trials for an a priori U(0,1) (i.e. Beta(1,1) distr.), we have $BMP(\theta)=5/10 = ½$

and for 9 successes in 10 trials for the same a priori distr., we have $BMP(\theta)=9/10$

Bayes Estimator: examples (2)

2. Let $X_1, \ldots, X_n$ be IID r.v. from $N(\theta, \sigma^2)$, with σ^2 known; $\theta \sim N(m, \tau^2)$ for m, τ known.

Then the a posteriori distr for θ: $N\left(\dfrac{n\frac{1}{\sigma^2}\overline{X} + \frac{1}{\tau^2}m}{n\frac{1}{\sigma^2} + \frac{1}{\tau^2}}, \dfrac{1}{n\frac{1}{\sigma^2} + \frac{1}{\tau^2}}\right)$

so

$$\hat{\theta}_B = \frac{n\frac{1}{\sigma^2}\overline{X} + \frac{1}{\tau^2}m}{n\frac{1}{\sigma^2} + \frac{1}{\tau^2}}$$

i.e. if we have sa sample of 5 obs 1.2; 1.7 ; 1.9 ; 2.1; 3.1 from distr. N(θ, 4) and the a priori distr is $\theta \sim N(1, 1)$, then

$\hat{\theta}_B$ = (5 /4 * 2 + 1)/(5/4 + 1) = 14/9 ≈ 1.56

and if the a priori distr were $\theta \sim N(3, 1)$, then

$\hat{\theta}_B$ = (5 /4 * 2 + 1*3)/(5/4 + 1) = 22/9 ≈ 2.44

BMP: examples (2)

2. Let $X_1, \ldots, X_n$ be IID r.v. from $N(\theta, \sigma^2)$, with σ^2 known; $\theta \sim N(m, \tau^2)$ for m, τ known.

Then the a posteriori distr for θ: $N\left(\dfrac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}, \dfrac{1}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right)$

so

$$BMP(\theta) = \frac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}$$

i.e. if we have sa sample of 5 obs 1.2; 1.7 ; 1.9 ; 2.1; 3.1 from distr. N(θ, 4) and the a priori distr is $\theta \sim N(1, 1)$, then

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and if the a priori distr were $\theta \sim N(3, 1)$, then

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Highest Posterior Density Credible Interval

A 1-α *HPD* (*Highest Posterior Density*) *credible interval (Bayesian Confidence Interval)* for parameter θ is a set A $\subseteq \Theta$ such that

$$A = \{\theta : \pi(\theta \mid \boldsymbol{x}) > k_\alpha\}$$

and

$$\Pi(A \mid \boldsymbol{x}) \geq 1 - \alpha$$

for k_α highest such that the second condition is fulfilled

The HPD credible interval has the intuitive property of inclusion which the frequentist CI does not have

HPD Credible Interval: example

Let $X_1, \ldots, X_n$ be IID r.v. from $N(\theta, \sigma^2)$, with σ^2 known; $\theta \sim N(m, \tau^2)$ for m, τ known.

Then the posterior distr. for θ: $N\left(\frac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}, \frac{1}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right)$

so for $\alpha = 0.05$ we get a HPD CI:

$$\left(\frac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}} - 1.96\cdot\frac{1}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}, \frac{n\frac{1}{\sigma^2}\overline{X}+\frac{1}{\tau^2}m}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}} + 1.96\cdot\frac{1}{n\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\right)$$

i.e. if we have a sample of 5 obs. from distr. N(θ, 4) with mean 2 and the a priori distr. is $\theta \sim N(1, 1)$, then due to the fact that $u_{0.975} \approx 1.96$ we have an HPD CI:

$$\approx \left(1.83 - 1.96\cdot \tfrac{1}{6}, 1.83 + 1.96\cdot\tfrac{1}{6}\right) = \left(1.50, 2.15\right)$$

Example problem

Let 0.38, 0.65, 0.72, 1.00 be independent realizations of a random variable from a uniform distribution over the interval $(0,\theta)$, where $\theta>0$ is an unknown parameter.

We initially assume that θ is uniformly distributed over the interval [1/2, 2].

Find the posterior distribution.

Find the Bayesian most probable estimator.

Find the Bayes estimator for a quadratic loss function.

Find the Bayes estimator for a modulus loss function.

Example exam problem

8. The time spent on a bus stop waiting for the bus (in minutes) has a uniform distribution over the interval $(0, \theta)$, where $\theta > 0$ is an unknown parameter. Two statisticians use the bus; they have observed the following waiting times (denoted. $X_1, \ldots, X_5$):

$$4,\ 3,\ 6,\ 1,\ 9.$$

   - Let us assume that the first statistician wants to verify the null hypothesis that $\theta = 8$ against the alternative that $\theta = 10$ with a test such that the critical region is $\{X_{5:5} > c\}$ for a constant c. For a test with significance level $\alpha = 0.1$, the constant c should be equal to:

     ....

     and the decision based on the observations is to REJECT /NO GROUNDS TO REJECT H_0 (underline the appropriate).
   - Let us assume that the second statistician, based on previous experience, supposes that the distribution of θ is given by a density

     $$\pi(\theta) = \frac{4}{6}\left(\frac{6}{\theta}\right)^5 \quad \text{for } \theta > 6,$$

     and zero otherwise. The Bayesian Most Probable estimate of θ, based on this *a priori* distribution and the observed sample, is equal to

     ....