

Mathematical Statistics

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HYPOTHESIS TESTING III:

LR TEST FOR COMPOSITE HYPOTHESES

EXAMPLES OF ONE-SAMPLE AND TWO-SAMPLE TESTS

Plan for today

1. LR test for composite hypotheses
 2. Examples of LR tests:
 - Model I: One- and two-sided tests for the mean in the normal model, σ^2 known
 - Model II: One- and two-sided tests for the mean in the normal model, σ^2 unknown
 - + One- and two-sided tests for the variance
 - Model III: Tests for the mean, large samples
 - Model IV: Tests for the fraction, large samples
 3. Asymptotic properties of the LR test
 4. Test randomization
 5. Parametric LR tests for two populations
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Testing simple hypotheses – reminder

We observe X . We want to test

$$H_0: \theta = \theta_0 \text{ against } H_1: \theta = \theta_1.$$

(two simple hypotheses)

We can write it as:

$$H_0: X \sim f_0 \text{ against } H_1: X \sim f_1,$$

where f_0 and f_1 are *densities* of distributions
defined by θ_0 and θ_1 (i.e. P_0 and P_1)



Likelihood ratio test for simple hypotheses. Neyman-Pearson Lemma – reminder

$H_0: X \sim f_0$ against $H_1: X \sim f_1$

Let
$$K^* = \left\{ x \in X : \frac{f_1(x)}{f_0(x)} > c \right\}$$

such that $P_0(K^*) = \alpha$ and $P_1(K^*) = 1 - \beta$

Then, for any $K \subseteq X$:

if $P_0(K) \leq \alpha$, then $P_1(K) \leq 1 - \beta$.

(i.e.: the test with critical region K^* is the most powerful test
for testing H_0 against H_1)



Likelihood ratio test for composite hypotheses

$X \sim P_\theta$, $\{P_\theta: \theta \in \Theta\}$ – family of distributions

We are testing $H_0: \theta \in \Theta_0$ against $H_1: \theta \in \Theta_1$

such that $\Theta_0 \cap \Theta_1 = \emptyset$, $\Theta_0 \cup \Theta_1 = \Theta$

Let

$H_0: X \sim f_0(\theta_0, \cdot)$ for some $\theta_0 \in \Theta_0$.

$H_1: X \sim f_1(\theta_1, \cdot)$ for some $\theta_1 \in \Theta_1$,

where f_0 and f_1 are densities (for $\theta \in \Theta_0$ and $\theta \in \Theta_1$, respectively)



Likelihood ratio test for composite hypotheses – cont.

Test statistic:
$$\lambda = \frac{\sup_{\theta_1 \in \Theta_1} f_1(\theta_1, X)}{\sup_{\theta_0 \in \Theta_0} f_0(\theta_0, X)}$$

or
$$\lambda = \frac{f_1(\hat{\theta}_1, X)}{f_0(\hat{\theta}_0, X)}$$

where $\hat{\theta}_0, \hat{\theta}_1$ are MLE of the null and alternative hypothesis models

We reject H_0 if $\lambda > c$ for a constant c
(determined according to significance level)



Likelihood ratio test for composite hypotheses – justification

Just like in the Neyman-Pearson Lemma, we compare the “highest chance of obtaining observation X , when the alternative is true” to the “highest chance of obtaining observation X , when the null is true”; we reject the null hypothesis in favor of the alternative if this ratio is very unfavorable for the null.



Likelihood ratio test for composite hypotheses – alternative version

Test statistic:
$$\tilde{\lambda} = \frac{\sup_{\theta \in \Theta} f(\theta, X)}{\sup_{\theta_0 \in \Theta_0} f_0(\theta_0, X)}$$

or
$$\tilde{\lambda} = \frac{f(\hat{\theta}, X)}{f_0(\hat{\theta}_0, X)}$$

where $\hat{\theta}, \hat{\theta}_0$ are the ML estimators for the model without restrictions and for the null model.

We reject H_0 if $\tilde{\lambda} > \tilde{c}$ for a constant $\tilde{c}$.



Likelihood ratio test for composite hypotheses

– properties

For some models with composite hypotheses the UMPT *does not exist* (so the LR test will not be UMP because there is no such test)

e.g. testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ if the family of distributions has a *monotonic LR property*, i.e. $f_1(x)/f_0(x)$ is an increasing function of a statistic $T(x)$ for any f_0 and f_1 corresponding to parameters $\theta_0 < \theta_1$.

In order to have UMPT for $H_0: \theta = \theta_0$ against $H_1: \theta < \theta_0$ we would need a critical region of the type $T(x) > c$, and to have a UMPT for $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ we would need a critical region of the type $T(x) < c$, so it is impossible to find a UMPT for $H_1: \theta \neq \theta_0$.



Likelihood ratio test: special cases

The exact form of the test depends on the distribution.

In many cases, finding the distribution is hard/complicated (in many such cases, we use the asymptotic properties of the LR test instead of precise formulae)



Notation

$x_{something}$ **always** means a quantile of rank something



Model I: comparing the mean

Normal model: $X_1, X_2, \dots, X_n$ are an IID sample from $N(\mu, \sigma^2)$, where σ^2 is **known**

$$H_0: \mu = \mu_0$$

Test statistic:

$$U = \frac{\bar{X} - \mu_0}{\sigma} \sqrt{n} \sim N(0,1)$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu > \mu_0$$

$$\text{critical region } K^* = \{x : U(x) > u_{1-\alpha}\}$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu < \mu_0$$

$$\text{critical region } K^* = \{x : U(x) < u_\alpha = -u_{1-\alpha}\}$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu \neq \mu_0$$

$$\text{critical region } K^* = \{x : |U(x)| > u_{1-\alpha/2}\}$$



Model I: example

Let $X_1, X_2, \dots, X_{10}$ be an IID sample from $N(\mu, 1^2)$:

-1.21 -1.37 0.51 0.37 -0.75 0.44 1.20 -0.96 -1.14 -1.40

Is $\mu = 0$? (for $\alpha = 0.05$)

In the sample: mean = -0.43, ~~variance = 0.92~~

Test statistic:
$$U = \frac{-0.43 - 0}{1} \sqrt{10} \approx -1.36$$

$H_0: \mu = 0$ against $H_1: \mu \neq 0$, $u_{0.975} \approx 1.96$ (p-value ≈ 0.172)

$H_0: \mu = 0$ against $H_1: \mu < 0$, $u_{0.05} \approx -1.64$ (p-value ≈ 0.086)

$H_0: \mu = 0$ against $H_1: \mu > 0$, $u_{0.95} \approx 1.64$ (p-value ≈ 0.914)

→ in none of these cases are there grounds to reject
 H_0 for $\alpha = 0.05$

→ but we would reject $H_0: \mu = 0$ in favor of $H_1: \mu < 0$ for $\alpha = 0.1$

Model II: comparing the mean

Normal model: $X_1, X_2, \dots, X_n$ are an IID sample from $N(\mu, \sigma^2)$, where σ^2 is **unknown**

$$H_0: \mu = \mu_0$$

Test statistic:

$$T = \frac{\bar{X} - \mu_0}{S} \sqrt{n} \sim t(n-1)$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu > \mu_0$$

$$\text{critical region } K^* = \{x : T(x) > t_{1-\alpha}(n-1)\}$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu < \mu_0$$

$$\text{critical region } K^* = \{x : T(x) < t_{\alpha}(n-1)\}$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu \neq \mu_0$$

$$\text{critical region } K^* = \{x : |T(x)| > t_{1-\alpha/2}(n-1)\}$$



$$t_{\alpha}(n-1) = -t_{1-\alpha}(n-1)$$

Model II: example (mean)

Let $X_1, X_2, \dots, X_{10}$ be an IID sample from $N(\mu, \sigma^2)$:

-1.21 -1.37 0.51 0.37 -0.75 0.44 1.20 -0.96 -1.14 -1.40

Is $\mu = 0$? (for $\alpha = 0.05$)

In the sample: mean = -0.43, variance = 0.92

Test statistic:
$$U = \frac{-0.43 - 0}{\sqrt{0.92}} \sqrt{10} \approx -1.42$$

$H_0: \mu = 0$ vs $H_1: \mu \neq 0$, $t_{0.975}(9) \approx 2.26$ (p-value ≈ 0.188)

$H_0: \mu = 0$ vs $H_1: \mu < 0$, $t_{0.05}(9) \approx -1.83$ (p-value ≈ 0.094)

$H_0: \mu = 0$ vs $H_1: \mu > 0$, $t_{0.95}(9) \approx 1.83$ (p-value ≈ 0.906)

→ in none of these cases are there grounds to reject H_0 for $\alpha = 0.05$

→ but we would reject $H_0: \mu = 0$ in favor of $H_1: \mu < 0$ for $\alpha = 0.1$

Model II: comparing the variance

Normal model: $X_1, X_2, \dots, X_n$ are an IID sample from $N(\mu, \sigma^2)$, where σ^2 is **unknown**

$$H_0: \sigma = \sigma_0$$

Test statistic:

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1)$$

$$H_0: \sigma = \sigma_0 \text{ against } H_1: \sigma > \sigma_0$$

$$\text{critical region } K^* = \{x : \chi^2(x) > \chi_{1-\alpha}^2(n-1)\}$$

$$H_0: \sigma = \sigma_0 \text{ against } H_1: \sigma < \sigma_0$$

$$\text{critical region } K^* = \{x : \chi^2(x) < \chi_{\alpha}^2(n-1)\}$$

$$H_0: \sigma = \sigma_0 \text{ against } H_1: \sigma \neq \sigma_0$$

$$\text{critical region } K^* = \{x : \chi^2(x) < \chi_{\alpha/2}^2(n-1)$$

$$\vee \chi^2(x) > \chi_{1-\alpha/2}^2(n-1)\}$$



Model II: example (variance)

Let $X_1, X_2, \dots, X_{10}$ be an IID sample from $N(\mu, \sigma^2)$:

-1.21 -1.37 0.51 0.37 -0.75 0.44 1.20 -0.96 -1.14 -1.40

Is $\sigma = 1$? (for $\alpha = 0.05$)

In the sample: variance = 0.92
Test statistic: $\chi^2 = \frac{9 \cdot 0.92}{1} \approx 8.28$

$H_0: \sigma = 1$ against $H_1: \sigma > 1$ $\chi_{0.95}^2 \approx 16.92$

$H_0: \sigma = 1$ against $H_1: \sigma < 1$ $\chi_{0.05}^2 \approx 3.33$

$H_0: \sigma = 1$ against $H_1: \sigma \neq 1$ $\chi_{0.025}^2 \approx 2.70; \chi_{0.975}^2 \approx 19.02$

→ in none of these cases are there grounds to reject
 H_0 (for $\alpha = 0.05$)



Model III: comparing the mean

Asymptotic model: $X_1, X_2, \dots, X_n$ are an IID sample from a distribution with mean μ and variance (unknown), n – large.

$$H_0: \mu = \mu_0$$

Test statistic:

$$T = \frac{\bar{X} - \mu_0}{S} \sqrt{n}$$

has, for large n , an approximate distribution $N(0,1)$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu > \mu_0$$

critical region

$$K^* = \{x : T(x) > u_{1-\alpha}\}$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu < \mu_0$$

critical region

$$K^* = \{x : T(x) < u_\alpha = -u_{1-\alpha}\}$$

$$H_0: \mu = \mu_0 \text{ against } H_1: \mu \neq \mu_0$$

critical region

$$K^* = \{x : |T(x)| > u_{1-\alpha/2}\}$$



Model IV: comparing the fraction

Asymptotic model: $X_1, X_2, \dots, X_n$ are an IID sample from a two-point distribution, n – large.

$$P_p(X = 1) = p = 1 - P_p(X = 0)$$

$$H_0: p = p_0 \quad U^* = \frac{\bar{X} - p_0}{\sqrt{p_0(1-p_0)}} \sqrt{n} = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)}} \sqrt{n}$$

Test statistic:

has an *approximate* distribution $N(0,1)$

$H_0: p = p_0$ against $H_1: p > p_0$

critical region $K^* = \{x : U(x) > u_{1-\alpha}\}$

$H_0: p = p_0$ against $H_1: p < p_0$

critical region $K^* = \{x : U(x) < u_\alpha = -u_{1-\alpha}\}$

~~$H_0: p = p_0$ against $H_1: p \neq p_0$~~

~~critical region $K^* = \{x : |U(x)| > u_{1-\alpha/2}\}$~~



Model IV: example

We toss a coin 400 times. We get 180 heads. Is the coin symmetric?

$$H_0: p = 1/2 \quad U^* = \frac{(180/400 - 1/2)}{\sqrt{1/2(1-1/2)}} \sqrt{400} = -2$$

for $\alpha = 0.05$ and $H_1: p \neq 1/2$ we have $u_{0.975} = 1.96 \rightarrow$ we reject H_0

for $\alpha = 0.05$ and $H_1: p < 1/2$ we have $u_{0.05} = -u_{0.95} = -1.64$

$\rightarrow$ we reject H_0

for $\alpha = 0.01$ and $H_1: p \neq 1/2$ we have $u_{0.995} = 2.58$

$\rightarrow$ we do not reject H_0

for $\alpha = 0.01$ and $H_1: p < 1/2$ we have $u_{0.01} = -u_{0.99} = -2.33$

$\rightarrow$ we do not reject H_0

p-value for $H_1: p \neq 1/2$: 0.044

p-value for $H_1: p < 1/2$: 0.022



Asymptotic properties of the LR test

We consider two nested models, we test

$H_0: h(\theta) = 0$ against $H_1: h(\theta) \neq 0$

Under the assumption that

- h is a nice function
- Θ is a d -dimensional set
- $\Theta_0 = \{\theta: h(\theta) = 0\}$ is a $d - p$ dimensional set

Theorem: If H_0 is true, then for $n \rightarrow \infty$ the distribution of the statistic $2\ln \tilde{\lambda}$ converges to a chi-squared distribution with p degrees of freedom



Asymptotic properties of the LR test – example

Exponential model: $X_1, X_2, \dots, X_n$ are an IID sample from $\text{Exp}(\theta)$.

We test $H_0: \theta = 1$ against $H_1: \theta \neq 1$

$$MLE(\theta) = \hat{\theta} = 1/\bar{X}$$

$$\tilde{\lambda} = \frac{\prod f_{\hat{\theta}}(x_i)}{\prod f_1(x_i)} = \frac{\frac{1}{\bar{X}^n} \exp(-\frac{1}{\bar{X}} \sum x_i)}{\exp(-\sum x_i)} = \frac{1}{\bar{X}^n} \exp(n(\bar{X} - 1))$$

then: $\tilde{\lambda} > \tilde{c} \Leftrightarrow 2\ln \tilde{\lambda} > 2\ln \tilde{c}$

from Theorem: $2\ln \tilde{\lambda} = 2n((\bar{X} - 1) - \ln \bar{X}) \xrightarrow{D} \chi^2(1)$

for a sign. level $\alpha = 0.05$ we have $\chi_{0.95}^2(1) \approx 3.84 \approx 2\ln \tilde{c}$

so we reject H_0 in favor of H_1 if $\tilde{\lambda} > e^{3.84/2}$



Test randomization

Sometimes, a test with a significance level exactly equal to α does not exist (e.g. for discrete random variables).

In such cases, we need *randomization*.

(The UMPT, if it exists, needs to be randomized).

eg. no of heads in 8 tosses, $H_0 : p = 1/2$, $H_1 : p < 1/2$, $\alpha=0.05$:

x_i	0	1	2	3	4	5	6	7	8
p_i	0.004	0.03	0.11	0.22	0.27	0.22	0.11	0.03	0.004
cum p_i	0.004	0.04	0.15	0.36	0.64	0.86	0.97	0.996	1.000

