

Problem sets 5 and 6
Estimator properties

1. The size of organisms from a given population may be described by a distribution with density $f_\beta(x) = \frac{x}{\beta^2} e^{-\frac{x}{\beta}}$ for $x > 0$ (and 0 otherwise). A sample of n organisms is drawn from the population. For the m.l.e. estimator of β (see Problem 3/Set 4), calculate the MSE, the bias and the estimator variance.

Hint: The expected value for this distribution is 2β , and the variance is $2\beta^2$.

2. A population is characterized by a density function of $f_\lambda(x) = \lambda e^{-\lambda x}$ for $x > 0$. Check that $\hat{v} = \frac{1}{2n} \sum_{i=1}^n X_i^2$ is an unbiased estimator of the population variance. What is the variance of this estimator? Calculate the MSE of the estimator.

Hint: The first four moments of the exponential distribution are equal to $\frac{1}{\lambda}, \frac{2}{\lambda^2}, \frac{6}{\lambda^3}, \frac{24}{\lambda^4}$.

3. 30 measurements of an unknown value μ were taken. Method A was used for the first 20 measurements, and the results – $X_1, X_2, \dots, X_{20}$ – are random variables from a normal distribution $N(\mu, 3^2)$. A different method B was used for the following 10 measurements, and the results – $X_{21}, \dots, X_{30}$ – are random variables from a normal distribution $N(\mu, 2^2)$. All measurements are independent. Let

$$\bar{X}_A = \frac{1}{20} \sum_{i=1}^{20} X_i, \quad \bar{X}_B = \frac{1}{10} \sum_{i=21}^{30} X_i.$$

Find a and b such that the estimator $\hat{\mu} = a\bar{X}_A + b\bar{X}_B$ is unbiased with minimum variance.

4. Assume that the amount (in USD) a random consumer is willing to spend yearly on water consumption follows a uniform distribution on the interval $[0, \theta]$, where $\theta > 0$ is an upper bound unknown to the researcher. The researcher surveys n independent individuals and records their yearly expenses $X_1, X_2, \dots, X_n$.

- (a) Find the m.l.e. of θ and calculate its bias and variance.
- (b) Construct an unbiased estimator on the basis of the m.l.e. of θ . Calculate the variance of this estimator.
- (c) For which value of a will $\hat{\theta}_a = \frac{a}{n} \sum_{i=1}^n X_i$ be an unbiased estimator of the parameter θ ? Determine the variance of this estimator.
- (d) Compare the three estimators above on the base of the MSE.
- (e) Construct the method of quantiles estimator for θ , based on the median. Verify whether this estimator is unbiased (you may assume that n is odd, i.e. that $n = 2l + 1$).

Hints: The m.l.e. of θ for a sample from a uniform distribution on $[0, \theta]$ is $\hat{\theta} = \max\{X_1, \dots, X_n\}$. The distribution of a k -th order statistic of a sample from a distribution with density $f(x)$ and cumulative distribution $F(x)$ has a density function $f_{X_{k:n}}(x) = n \binom{n-1}{k-1} f(x) F(x)^{k-1} (1 - F(x))^{n-k}$.

5. Let $X_1, X_2, \dots, X_n$ denote the prices (in EUR) of a given article in different shops. We assume these observations are independent, from a normal distribution with unknown mean μ and variance

σ^2 . Previous research suggests that the average price level is around 50, so the researcher uses the following “conservative” estimator of the parameter μ :

$$\hat{\mu} = \frac{50 + \bar{X}}{2}.$$

- (a) Verify whether this estimator is unbiased. Calculate the MSE.
 - (b) Verify whether this estimator is consistent.
6. Let $\theta \in (0, 1)$ denote the probability that a random client entering a shop will buy a box of chocolates. Let $X_1, X_2, \dots, X_{2n}$ denote the outcomes (1 – purchase, 0 – otherwise) for $2n$ independent consumers ($2n > 20$).
- (a) Denote by $\hat{\theta}_{MLE}$ the m.l.e. of θ on the base of the sample $X_1, X_2, \dots, X_{2n}$, by $\hat{\theta}_2$ the m.l.e. on the base of odd observations only, and by $\hat{\theta}_3$ the m.l.e. on the base of the first 20 observations (i.e. $X_1, \dots, X_{20}$).
 - (b) Check that these estimators are unbiased.
 - (c) Verify whether these estimators are efficient.
 - (d) Verify whether these estimators are consistent.