**1. Probability Theory.**

1. Random vector.
2. Expected value of random vector.
3. Conditional expected value of random vector.
4. Variance-covariance matrix of random vector.
5. Properties of normal distribution, $chi - square$ distribution, $t$ distribution, $F$ distribution.

**1.1 Exercises: expected value, variance.**

1. Show that $Cov(x_i x_j) = E(x_i, x_j) - E(x_i)E(x_j)$.
2. Show that if $E(x_i) = 0$ than $Var(x_i) = E(x_i^2)$.
3. Which of matrices:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 4 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

can be variance-covariance matrices?

4. Show, that for any random vector $\varepsilon$, non-random vector $a$ and non-random matrix $B$:

$$E(a + B\varepsilon) = a + BE(\varepsilon)$$

$$Var(a + B\varepsilon) = BVar(\varepsilon)B'$$

5. We have random vector $x$, $E(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, Var(x) = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. Compute expected value and variance of $y = \begin{bmatrix} x_1 + 2x_2 + 5 \\ x_1 + x_2 + 1 \end{bmatrix}$.
6. We have random vector $x$, $E(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, Var(x) = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$.

   Compute:

   a. Standard deviation of $x_1, x_2$.
   b. Correlation between $x_1, x_2$.
   c. Expected value and variance of $y = 5 + x_1 + 2x_2$.
7. Proof that for any random matrix $A$ :$E[tr(A)] = tr[E(A)]$.
8. Assume that $E(x) > 0$. What is the relation between $E(x), E\left(\frac{1}{x}\right)$?

   Hint: use Jensen theorem.

9. Assume that $y, x$ are random variables, $E\left(\frac{y}{x}|x\right)$=?
10. $E(x) = 2, \quad E(y|x) = 1 + 2x. E(y)$ =?

**2. Normal distribution.**

1. What is the distribution of $v = a + B\varepsilon$, if $\varepsilon \sim N(0, \Sigma)$ ?

2. Show that for k-dimension random vector $\varepsilon \sim N(0, \Sigma)$ ,
quadratic form $\varepsilon' \Sigma^{-1} \varepsilon \sim chi - squre(k)$.
3. We have random vector $x \sim N(\mu, \Sigma)$, where $\mu = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. What is the distribution of random variable $v = x_1 + 2x_2 + x_3$?
4. We have random vector $x \sim N(\mu, \Sigma)$, where $\mu = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$.
Show that $v = \begin{bmatrix} x_1 - x_2 - 1 \\ -x_1 + 2x_2 + 4 \end{bmatrix}$ has distribution $v \sim N(0, I)$.
Proof that $(x_1 - x_2 - 1)^2 + (-x_1 + 2x_2 + 4)^2 \sim chi - square(2)$.
Show the same for $\Sigma^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

### 3. Statistics.

1. Estimator.
2. Unbiased estimator.
3. Variance of estimator and efficiency.
4. Confidence interval.
5. Statistical hypothesis testing, critical values, p-value.

### 3.1 Exercises.

1. Show, that for two estimators $\hat{\theta}, \tilde{\theta}$ of parameter vector $\theta$ with variances $\widehat{\Sigma}, \widetilde{\Sigma}$ and the positive-definite difference $\widehat{\Sigma} - \widetilde{\Sigma}$ :
$$Var(\delta'\hat{\theta}) > Var(\delta'\tilde{\theta})$$
2. We have random variables $y_1, y_2$.
$$E(y_1) = \theta, \qquad E(y_2) = \frac{1}{2}\theta, \qquad Var(y_1) = 3\sigma^2, \qquad Var(y_2) = \sigma^2, Cov(y_1, y_2) = \sigma^2$$
   a. What are the conditions on $a_1, a_2$ for estimator $\hat{\theta} = a_1 y_1 + a_2 y_2$ to be unbiased.
   b. Find $a_1, a_2$ for estimator $\hat{\theta}$ to have the smallest variance and be unbiased.
   c. For $y_1, y_2$ having normal distribution find the distribution of $\hat{\theta}$.
3. We have $n$-dimension vector $x$. Elements if this vector have the same expected value $\mu$ and variance $\sigma^2$ and are non-correlated.
   a. Find variance-covariance matrix of vector $x$.
   b. Proof that $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is unbiased estimator of $\mu$.
   c. Show that variance of $\bar{x}$ decreases when N increases.
4. We have estimator $\hat{\theta}$ of parameter $\theta$ and $se(\hat{\theta})$. We know that $\frac{\hat{\theta}-\theta}{se(\hat{\theta})} \sim t_s$, where $s$ is the number of observations. For $\hat{\theta} = 1, se(\hat{\theta}) = 0.5$, s=10:
   a. Build 95% confidence interval for $\hat{\theta}$.
   b. What will happen with the confidence interval if we will change 95% to 90%?
   c. What will probably happen with confidence interval if n increases?
   d. Verify hypothesis: $H_0 = 0, \alpha = 0.5$.
   Hint: $t_{10}(2) = 0.07$ (value of cumulative distribution function)