## MATHEMATICAL ANALYSIS

1. Derivative

- Lets $f(x_1, \ldots, x_n): R^n \to R$ (scalar function)
  The derivative of a scalar function with respect to vector $x$ is the vector of partial derivatives.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

- Lets $f(x_1, \ldots, x_n): R^n \to R^m$ (vector function)

$$f(x_1, \ldots, x_n) = \left(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\right)$$

  where $f_i$ are scalar functions

  example: $f: R^3 \to R^2$ $f(x_1, x_2, x_3) = (x_1 + x_2, x_3)$

  $f_1(x_1, x_2, x_3) = x_1 + x_2$

  $f_2(x_1, x_2, x_3) = x_3$

  The derivative of a vector function with respect to vector $x$ is a matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

2. $\alpha, \beta$ – column vectors
   $\frac{\partial \alpha\prime\beta}{\partial \beta} = \alpha$ , $\frac{\partial \alpha\prime\beta}{\partial \beta\prime} = \alpha'$

$$\beta' = [\beta_1, \ldots, \beta_k]$$
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   We define scalar function: $f(\beta_1, \ldots, \beta_k) = \alpha'\beta = \alpha_1\beta_1 + \cdots + \alpha_k\beta_k$

$$\frac{\partial f}{\partial \beta_i} = \alpha_i \quad \text{so} \quad \frac{\partial \alpha\prime\beta}{\partial \beta} = \frac{\partial f}{\partial \beta} = \begin{bmatrix} \frac{\partial f}{\partial \beta_1} \\ \vdots \\ \frac{\partial f}{\partial \beta_k} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = \alpha$$

$$\frac{\partial \alpha\prime\beta}{\partial \beta\prime} = \begin{bmatrix} \frac{\partial \alpha\prime\beta}{\partial \beta_1} & \cdots & \frac{\partial \alpha\prime\beta}{\partial \beta_k} \end{bmatrix} = [\alpha_1 \quad \cdots \quad \alpha_k] = \alpha'$$

## EXERCISES

1. Find gradient and hessian of function $y = 2x_1^2 + 3x_2^2 + 5x_1x_2 - 4$. Find extreme of this function and point its type.

Gradient $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 + 5x_2 \\ 6x_2 + 5x_1 \end{bmatrix}$

Hessian $H(x) = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1 \partial x_1} & \frac{\partial^2 y}{\partial x_1 \partial x_2} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2 \partial x_2} \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}$

Extreme $\begin{cases} 4x_1 + 5x_2 = 0 \\ 6x_2 + 5x_1 = 0 \end{cases} \rightarrow \ x_1 = 0, x_2 = 0$

$d_1 = 4$

$\det H(x) = 4 \times 6 - 5 \times 5 = -1$

Hessian is non-define.

2. Find extreme of function $y = x_1^2 + 4x_2^2 + x_1x_2 - 1$ and point its type. Find extreme for the same function with extra condition $x_2 - 2x_1 = 1$.

Gradient $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 8x_2 + x_1 \end{bmatrix}$

Hessian $H(x) = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1 \partial x_1} & \frac{\partial^2 y}{\partial x_1 \partial x_2} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2 \partial x_2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}$

Extreme $\begin{cases} 2x_1 + x_2 = 0 \\ 8x_2 + x_1 = 0 \end{cases} \rightarrow \ x_1 = 0, x_2 = 0$

$d_1 = 2$

$\det H(x) = 2 \times 8 - 1 \times 1 = 15$

So Hessian is positive-define, so we have found minimum.

Condition $x_2 - 2x_1 = 1$

Lagrange function $y = x_1^2 + 4x_2^2 + x_1x_2 - 1 + \lambda(x_2 - 2x_1 - 1)$

$\frac{\partial y}{\partial x_1} = 2x_1 + x_2 - 2\lambda = 0$

$\frac{\partial y}{\partial x_2} = 8x_2 + x_1 + \lambda = 0$

$\frac{\partial y}{\partial \lambda} = x_2 - 2x_1 - 1 = 0$

We get $x_1 = -\frac{17}{38}, \quad x_2 = \frac{4}{38}$

Goal function

$y(0{,}0) = 0 + 0 + 0 - 1 = -1$

$y\left(-\frac{17}{38}, \frac{4}{38}\right) = -\frac{1159}{1444}$