I ALGEBRA

Main definitions

1. A matrix, operations on matrices: adding, multiplication, transposition.
2. Determinant of a matrix (definition and properties).
3. Quadratic forms (positive define matrix).
4. Trace of a matrix (definition and properties).
5. Idempotent matrix.

EXERCISES.

1. The following matrices are given:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Compute:
$2A$, B', $A + B'$, AB, $|AB| = \det(AB)$.

Explain, why it is not possible to compute AB', $A + B$.

2. The following matrices are given:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

Show that matrix A is symmetric.
Show that $AB \neq BA$
Show that A, AB are singular.

3. Expand expressions (assume, that A and B are invertible): $(A + B)(C + D)'$, $(AB)^{-1}(B)^{-1}$, $(BA)^{-1}(B)^{-1}$, $A(A + B)^{-1}$

4. Show that for any invertible A: $(A^{-1})' = (A')^{-1}$

5. Proof (from definition), that matrix $X'X$ is non-negative define.

6. Proof (from definition of linear independence), that matrix $X'X$ is non-singular when columns of matrix matrix X are linearly independent.

7. (*)

8. (*)

9. Proof, that for matrices A and B (assume proper sizes) $(AB)' = B'A'$.

10. Proof, that for trace of a matrix $tr(A + B) = tr(A) + tr(B)$, $tr(AB) = tr(BA)$.

11. Proof, that for idempotent P, $M = I - P$ is also idempotent, and that $MP = 0$.

12. Proof, that for any matrix A (such that $A'A$ is non-singular) $P = A(A'A)^{-1}A'$ is idempotent.

13. Proof, that $M = I - n^{-1}ll'$ is an idempotent matrix and that $l'M = 0$.

$$l = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Compute $tr(M)$.

II MATHEMATICAL ANALYSIS

Main definitions

1. Derivatives of scalar and vector functions with respect to a vector.
2. Proof, that for a column vectors α and β: $\frac{\partial \alpha'\beta}{\partial \beta} = \alpha, \frac{\partial \alpha'\beta}{\partial \beta'} = \alpha'$
3. Proof, that $\frac{\partial A\beta}{\partial \beta\prime} = A$ and $\frac{\partial \beta\prime A}{\partial \beta} = A$
4. Proof, that $\frac{\partial \beta' A\beta}{\partial \beta} = 2A\beta$
5. What should be the value of gradient of continuous and differentiable function $f(\beta)$ at point β^*, to provide that function has maximum at point β^*.
6. What is the characteristic feature of matrix of second derivatives?
7. How can we recognize, based on properties of matrix of second derivatives, that extreme of function of several variables is maximum.

EXERCISES.

1. Find gradient and Hessian of a function $y = 2x_1^2 + 3x_2^2 + 5x_1x_2 - 4$. Find extreme of this function and point its type.
2. Find extreme of function $y = x_1^2 + 4x_2^2 + x_1x_2 - 1$ and point its type. Find extreme for the same function with extra condition $x_2 - 2x_1 = 1$ (use Lagrange function). Compere extreme with and without extra condition.
3. The following values were found: $g^* = \max_{x_1x_2} g(x_1, x_2)$ and $g^{**} = \max_{x_1} g(x_1 0)$. Compare g^* and g^{**}.
4. The following values were found: $g^* = \max_{x_1x_2} g(x)$ (maximum with extra condition) and $g^{**} = \max_{x_1x_2} g(x)$ (maximum without extra condition). Compare g^* and g^{**}.